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Answers to version A of Exam #1

2/27/98

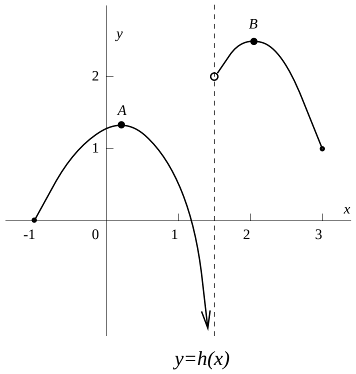
(12) 1. Below is a graph of $y = h(x)$, a function whose domain is $-1 \le x < 1.5$ and $1.5 < x \le 3$. Use this graph to answer the questions below.

a) Which of the following limits exist? If they exist, what are their values? If they do not exist, explain why.

$\lim_{x\to 1.5^-} h(x)$ **Answer** The graph seems to head *down* as $x \to 1.5$ from the left, so this limit is $-\infty$.

$\lim_{x\to 1.5^+} h(x)$ **Answer** The graph seems to go towards the "empty point" as $x \to 1.5$ from the right. The "empty point" seems to be at $(1.5, 2)$, so this limit is 2.

$\lim_{x\to 1.5} h(x)$ **Answer** This two-sided limit does not exist, since if it existed the two one-sided limits above would have to agree.

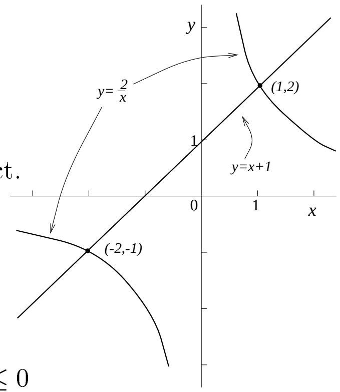


b) Are there any values of x where $h'(x) = 0$? If there are, what are the approximate values of $h(x)$ at these points? **Answer** Yes, there are values of x where $h'(x) = 0$. Those values of x occur at points on the curve where the curve has a horizontal tangent line. There seem to be two such points, indicated by a • in the picture, and labeled A and B. A has approximate coordinates $(.25, 1.25)$ and B has approximate coordinates $(2.1, 2.3)$. The approximate values of $h(x)$ at the two points seem to be 1.25 and 2.3.

(14) 2. The hyperbola $y = \dfrac{2}{x}$ and the line $y = x + 1$ intersect in two points.

- Use algebra to find the coordinates of these points.
- Sketch both curves on the indicated axes.
- Label the curves and label the coordinates of the points where they intersect.

Answer Certainly $\frac{2}{x} = x + 1$ because the y coordinates must be equal. Therefore $2 = x^2 + x$ and then $x^2 + x - 2 = 0$. This quadratic factors easily: $(x + 2)(x - 1) = 0$, and we read the roots: $x = 1$ or $x = -2$. When $x = 1$ then $y = 2$ and when $x = -2$, then $y = -1$.



(16) 3. Suppose that the function F is described by $F(x) = \begin{cases} e^x, & \text{if } -1 \le x \le 0 \\ 3x + A, & \text{if } 0 < x \le 2 \end{cases}$.

a) Find A so that F is continuous for all numbers in its domain. Briefly explain your answer.

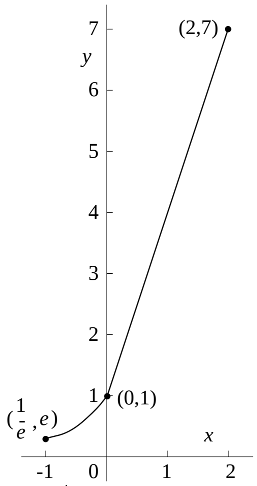
Answer The only discontinuity can occur at 0. The left- and right-hand limits of F at 0 must agree with F's value at 0. $F(0) = e^0 = 1$. $\lim_{x\to 0^-} F(x) = \lim_{x\to 1^-} e^x = e^0 = 1$. $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} 3x + A = 0 + A = A$. So we have $A = 1$.

b) Graph $y = F(x)$ on the axes given for the value of A found in a).

Comment Dots have been added for emphasis at the "edges" of each formula's interval of definition. The points have been labeled which is helpful in part d).

c) With the value of A found in a), is F differentiable at $x = 0$? Briefly explain your answer.

Answer F is *not* differentiable at $x = 0$. F has a corner or a "kink" there, and doesn't have a nice tangent line. From the left, the tangent line's slope would be the derivative of e^x at 0, which is 1, while from the right the tangent line's slope would have to be the derivative of $3x + 1$ at 0, which is 3. F is not differentiable at 0 because $3 \neq 1$.



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d) Suppose G is the function inverse to F with the value of A found in a). Describe G using one or more formulas: be as precise as you can with G's algebraic description and with its domain.
Answer The inverse to e^x is certainly $\ln x$. For the formula $y = 3x + 1$, we solve for x and get $x = \frac{1}{3}(y - 1)$ which gives a formula for G in the "other" interval. So: $G(x) = \begin{cases} \ln x, & \text{if } \frac{1}{e} \le x \le 1 \\ \frac{1}{3}(x-1), & \text{if } 1 < x \le 7 \end{cases}$. Note that it doesn't matter which formula "owns" the domain value 1, since $G(1)$ will be the same.

(20) 4. Evaluate the indicated limits exactly whenever possible. Give evidence to support your answers.

a) $\lim\limits_{x\to 3} \dfrac{x^2 - 5x + 6}{x - 3}$ **Answer** Since $x^2 - 5x + 6 = (x-3)(x-2)$, the quotient is $x - 2$ which has limit 1 as $x \to 3$.

b) $\lim\limits_{x\to 1} \dfrac{\frac{1}{x} - 1}{x - 1}$ **Answer** $\frac{1}{x} - 1 = \frac{(1-x)}{x}$ so the quotient becomes $\frac{\frac{(1-x)}{x}}{x-1} = \frac{-1}{x}$ which has limit -1 as $x \to 1$.

c) $\lim\limits_{x\to 0^-} \dfrac{3x + \cos x}{x}$ **Answer** The top becomes $3 \cdot 0 + \cos(0) = 1$ as $x \to 0$ and the bottom becomes a small negative number as $x \to 0^-$. Therefore the quotient becomes large negative, and the answer is $-\infty$.

d) $\lim\limits_{x\to 2} \dfrac{x}{x^2 - 4}$ **Answer** The top becomes 2 as $x \to 2$ but the bottom becomes a small negative number if $x \to 2^-$ and a small positive number if $x \to 2^+$. From the left the limit is $-\infty$ and from the right the limit is $+\infty$, so the two-sided limit requested here does not exist.

(11) 5. a) Write the definition of derivative as a limit and *use this definition* to find the derivative of $f(x) = 5\sqrt{x}$.
Answer $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$. We simplify this with our specific f when $h \ne 0$: $\frac{f(x+h)-f(x)}{h} = \frac{5\sqrt{x+h}-5\sqrt{x}}{h} = 5\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right) = 5\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\cdot\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = 5\left(\frac{(x+h)-x}{h(\sqrt{x+h}+\sqrt{x})}\right) = 5\left(\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right) = \frac{5}{\sqrt{x+h}+\sqrt{x}}$.
This last algebraic form allows us to find the limit: $f'(x) = \lim\limits_{h\to 0} \frac{5}{\sqrt{x+h}+\sqrt{x}} = \frac{5}{2\sqrt{x}}$.

b) Use your answer to a) to find an equation for the line tangent to the curve $y = 5\sqrt{x}$ when $x = 4$.
Answer The slope of the tangent line when $x = 4$ is $f'(4) = \frac{5}{2\sqrt{4}} = \frac{5}{4}$ and one point of the tangent line is the point $(2, f(2))$ which is $(2, 2)$. Therefore an equation for the line is $y - 2 = \frac{5}{4}(x - 2)$. This is a fine answer. It can be converted to slope-intercept form: $y = \frac{5}{4}x - \frac{1}{2}$ (but it doesn't have to be).

(17) 6. a) If $f(x) = -7x^8 + 3x^3 + 14$, what is $f'(x)$? **Answer** $f'(x) = -7(8x^7) + 3(3x^2) + 0$.

b) If $g(x) = \dfrac{x^2 - 2e^x}{5x - 3}$, what is $g'(x)$? **Answer** $g'(x) = \dfrac{(2x - 2e^x)(5x-3) - 5(x^2 - 2e^x)}{(5x-3)^2}$.

c) If $h(x) = \sqrt{x^3 - x + 1}$ what is $h'(x)$? **Answer** $h'(x) = \frac{1}{2}(x^3 - x + 1)^{-1/2}(3x^2 - 1)$.

d) Suppose that $j(x) = k(\sin x)$ (this is a composition). The following is known about k:

$$k(0) = 3 \ \&\ k'(0) = -4 \ \&\ k''(0) = 5$$

Compute $j(\pi)$ and $j'(\pi)$ and $j''(\pi)$. The answers should be exact integers in each case.
Answer $j(\pi) = k(\sin(\pi)) = k(0) = \mathbf{3}$. $j'(x) = k'(\sin x)\cos x$ by the chain rule, so $j'(\pi) = k'(\sin(\pi))\cos(\pi) = k'(0)\cdot(-1) = \mathbf{4}$. Now use the chain rule *and* the product rule to get $k''(x) = k''(\sin x)(\cos x)^2 + k'(\sin x)(-\sin x)$, so $j''(\pi) = k''(\sin(\pi))(\cos(\pi))^2 + k'(\sin(\pi))(-\sin(\pi)) = k''(0)\cdot(-1)^2 + k'(0)\cdot 0 = \mathbf{5}$.

(10) 7. A line passing through the origin is tangent to the graph of $y = Cx^2 + D$ at the point $(2, 16)$. Use methods of calculus to find C and D.
Answer The tangent line has slope $2Cx$ (the derivative). At $x = 2$ this is $4C$, which should be the same as $\frac{16-0}{2-0} = 8$ (the quotient is $\frac{\triangle y}{\triangle x}$). Therefore $C = 2$. Since $(2, 16)$ is on the curve $y = Cx^2 + D$, we know $16 = C(2^2) + D$. But if $C = 2$, then $D = 8$.