640:135:01-03 **Answers to version B of Exam #2** **4/10/98**

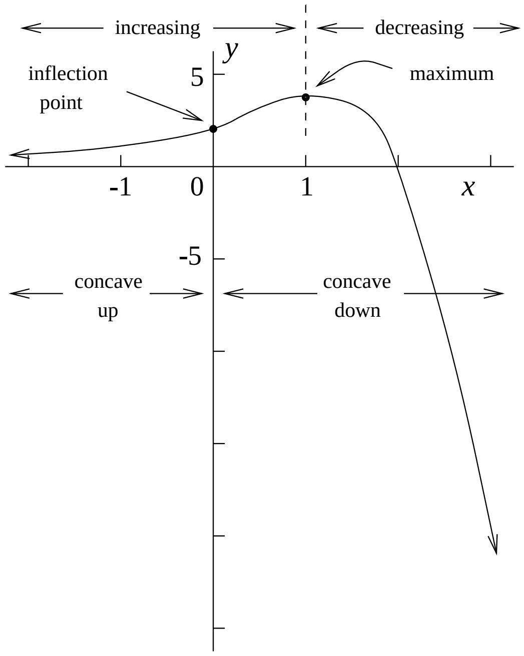
(9) 1. The equation $(4-x)y^2 = x^3$ defines a curve. Write an equation for a line tangent to this curve at the point $(2,2)$. **Answer** Differentiate the equation with respect to x nto get $-y^2 + 2(4-x)yy' = 3x^2$ so $y' = \frac{3x^2+y^2}{2(4-x)y}$. At $(2,2)$, $y' = \frac{3\cdot 2^2+2^2}{2(4-2)2} = \frac{16}{8} = 2$, so an equation is $y - 2 = 2(x-2)$.

(10) 2. Use calculus to locate the maximum and minimum values of the function $f(x) = x^3 - 12x$ on the closed interval $[0,4]$. **Answer** The extrema will occur either at endpoints (0 and 4) or at critical numbers in the interval. Since $f'(x) = 3x^2 - 12$, critical numbers are ± 2 and the relevant critical number is 2. The simplest way to get the extrema is to evaluate: $f(0) = 0$ and $f(2) = 8 - 12(2) = -16$ and $f(4) = 64 - 12(4) = 16$. So the maximum value of f in $[0,4]$ is 16 and the minimum value of f in $[0,4]$ is -16.

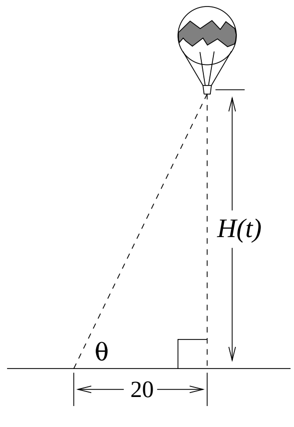
(21) 3. Suppose that $F(x) = e^x(2-x)$.
a) Where is $F(x) = 0$? **Answer** The exponential function is never 0, so $F(x) = 0$ only when $x = 2$.
b) Compute $F'(x)$. Where is $F'(x) = 0$? **Answer** $F'(x) = e^x(2-x) + e^x(-1) = e^x(1-x)$. Therefore $F'(x) = 0$ only when $x = 1$.
c) Compute $F''(x)$. Where is $F''(x) = 0$? **Answer** $F''(x) = e^x(1-x) + e^x(-1) = e^x(-x)$. Therefore $f''(x) = 0$ only when $x = 0$.
d) Sketch a graph of $y = F(x)$. Use calculus to identify the following features on your graph. Describe each of these algebraically and label each one clearly on your graph.

- Intervals where F is increasing and where it is decreasing. **Answer** F is increasing on $(-\infty, 1]$ and F is decreasing on $[1, +\infty)$.
- Intervals where F is concave up and where it is concave down. **Answer** F is concave up on $(-\infty, 0]$ and F is concave down on $[0, +\infty)$.
- The coordinates of any maxima or minima of F. **Answer** F has a maximum at $(1, F(1)) = (1, e)$.
- The coordinates of any inflection points of F. **Answer** F has an inflection point at $(0, F(0)) = (0, 2)$.

Note the difference in scale on the vertical and horizontal axes drawn below. Remember to *label* your graph!

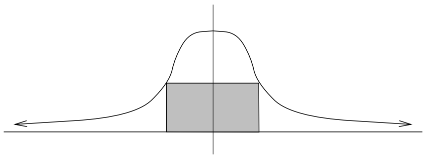


(10) 4. An observer is standing 20 feet away across level ground from the site of a balloon launch. The balloon travels straight up. Suppose that at time t, the bottom of the balloon's gondola is $H(t)$ feet from the ground.
a) Write an equation relating $H(t)$ and θ, the angle seen by the observer between the ground and the bottom of the gondola. You will need to use a trigonometric function. **Answer** $\tan\theta = \frac{H(t)}{20}$
b) At a certain time, the bottom of the gondola is 50 feet off the ground and is rising at 6 feet per second. At that time, what is the angle θ defined in part a) and what is the rate of change of that angle? **Answer** At the "certain time", $\tan\theta = \frac{50}{20} = 2.5$. Then $\theta = \arctan 2.5 \approx 1.19$ radians. Differentiating the equation in a) gives: $(\sec(\theta))^2\,\theta' = \frac{H'(t)}{20}$. At the "certain time" this becomes: $(\sec(1.19))^2\,\theta' = \frac{6}{20} = .3$, so $\theta' = \frac{.3}{(\sec(1.19))^2}$ which is a fine answer. Computing further, $\theta' = \frac{.3}{7.25} \approx .0414$ radians/sec.



(14) 5. Find the dimensions of the rectangle with *largest* area which has one side on the x-axis and two corners on the curve $y = \dfrac{1}{3+x^2}$. Explain briefly why you have found the largest such rectangle.
Answer The horizontal dimension of the rectangle is $2x$ and its vertical dimension is $\frac{1}{3+x^2}$. Therefore the area of the rectangle is $A = \frac{2x}{3+x^2}$. The domain of interest for this function is $0 < x < \infty$. Extrema are

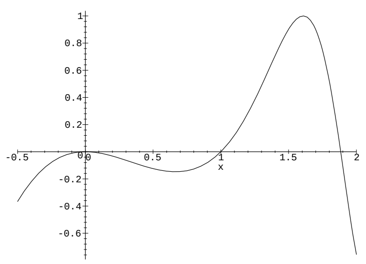
expected where $A' = 0$ so a computation of A' will be useful: $A' = \frac{2(3+x^2)-2x\cdot 2x}{(3+x^2)^2} = \frac{6-2x^2}{(3+x^2)^2}$. $A' = 0$ in our domain when $x = \sqrt{3}$. The dimensions of the rectangle for this value of x are $2\sqrt{3}$ and $\frac{1}{6}$. Why do these values represent a maximum? Here are some possible reasons (just one needs to be given!):



Reason #1 $\lim\limits_{x\to 0^+} A = 0$ and $\lim\limits_{x\to +\infty} A = 0$ follow easily from the formula for A. So the positive area found when $x = \sqrt{3}$ must represent a maximum. **Reason #2** Since $A' = \frac{6-2x^2}{(3+x^2)^2}$, when x is to the left of $\sqrt{3}$ (say, x near 0) $A' > 0$ and when x is very large positive, $A' < 0$. Therefore A increases to the left of $\sqrt{3}$ and decreases to the right of $\sqrt{3}$. So $\sqrt{3}$ must represent a maximum. **Reason #3** A tedious computation shows that $A''(\sqrt{3}) = -\frac{\sqrt{3}}{9}$. Thus A is concave down and must have a maximum at $\sqrt{3}$.

(12) 6. Suppose $g(x) = \sqrt{1+3x^2}$. Then $g(1) = 2$.
a) Use the differential or tangent line approximation to find an approximate value for $g(1.04)$. **Answer** $g(a+h) \approx g(a) + g'(a)h$. Here $a = 1$ and $h = .04$ and $g'(x) = \frac{1}{2}(1+3x^2)^{-1/2}6x = \frac{3x}{\sqrt{1+3x^2}}$. Therefore $g(1.04) \approx g(1) + g'(1)(.04) = \sqrt{1+3\cdot 1^2} + \frac{3\cdot 1}{\sqrt{1+3\cdot 1^2}}(.04) = 2 + \frac{3}{2}(.04) = 2.06$.
b) The true value of $g(1.04)$ is 2.06029 12415 to ten-decimal place accuracy. Use calculus to explain why the approximation you found in a) is greater than or less than the true value. **Answer** The answer in a) is *less than* the true value. The error is measured by the value of the second derivative at 1: $g''(1)$. If $g''(1) > 0$ then the graph is concave up at 1, explaining the underestimation. Since $g'(x) = \frac{3x}{\sqrt{1+3x^2}}$, $g''(x) = \frac{3\sqrt{1+3x^2} - 3x\frac{3x}{\sqrt{1+3x^2}}}{(\sqrt{1+3x^2})^2}$. No "simplification" of this will be attempted, since only the value of $g''(1)$ is needed. Here it is: $\frac{3\sqrt{1+3\cdot 1^2} - 3\frac{3}{\sqrt{1+3\cdot 1^2}}}{(\sqrt{1+3\cdot 1^2})^2} = \frac{3\cdot 2 - 3\cdot\frac{3}{2}}{2^2} = \frac{6-4.5}{4} = \frac{1.5}{4} = .375$. Notice that $.375 > 0$, verifying the desired concavity of g at 1.

(24) 7. The function $H(x)$ has domain $-.5 \le x \le 2$ and $H'(x) = \sin(x^3 - x^2)$. A graph of the derivative of H is shown below. Use the graph of H' to answer the questions below. Don't compute H'' algebraically, and don't try to get a formula for H.
a) Does H have any critical numbers? Write the approximate value of each critical number. **Answer** Yes. $x = 0$, 1, and 1.9.
b) Which of the critical numbers are relative maxima, which are relative minima, and which are neither? Give brief reasons for each of your answers. **Answer** Near 0 both left and right, $H' < 0$. Therefore 0 is neither a relative maximum nor a relative minimum.



The graph of $H'(x)$, the **derivative** of $H(x)$.

Since H' changes sign from $-$ to $+$ at 1, H must have a relative minimum at 1. Since H' changes sign from $+$ to $-$ at 1.9, H must have a relative maximum at 1.9. An alternative approach would be to consider the sign of H'' at 1 and 1.9, and decide from that what the concavity of H is at those points. Then that information could be used to determine the nature of those critical numbers.
c) In which intervals is H concave up? In which intervals is H concave down? **Answer** H is concave up where H' is increasing: $[-.5, 0]$ and $[.7, 1.6]$, and is concave down where H' is decreasing: $[0, .7]$ and $[1.6, 2]$.
d) Does H have any inflection points? Write the approximate first (x) coordinate of each inflection point. **Answer** Yes, H has inflection points where its concavity changes: at $x = 0$, .7, and 1.6.
e) Sketch a possible graph of $y = H(x)$ using all the information you have obtained in previous parts of this problem.
Label each relative minimum with an **m** and each relative maximum with an **M** and each inflection point with an **I**.