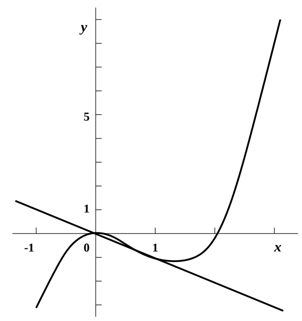
640:151:04–06 **Answers to version A of the first exam** **10/14/2003**

(12) 1. a) State the formal definition of the derivative, $f'(x)$, of the function $f(x)$.
Answer $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
b) Use your answer to a) combined with algebraic manipulation and standard properties of limits to compute the derivative of $f(x) = \sqrt{5x+3}$.
Answer If $f(x) = \sqrt{5x+3}$, $f(x+h) = \sqrt{5(x+h)+3}$, and $\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{5(x+h)+3}-\sqrt{5x+3}}{h}$. Multiply top and bottom by $\sqrt{5(x+h)+3}+\sqrt{5x+3}$. On top the result will be $(\sqrt{5(x+h)+3})^2 - (\sqrt{5x+3})^2 = 5(x+h)+3-(5x+3) = 5h$. The bottom becomes $h(\sqrt{5(x+h)+3}+\sqrt{5x+3})$. So the difference quotient $\frac{f(x+h)-f(x)}{h} = \frac{5h}{h(\sqrt{5(x+h)+3}+\sqrt{5x+3})} = \frac{5}{\sqrt{5(x+h)+3}+\sqrt{5x+3}}$. As $h \to 0$, $\sqrt{5(x+h)+3} \to \sqrt{5x+3}$ and the limit of the difference quotient is $\frac{5}{2\sqrt{5x+3}}$, which is $f'(x)$.

(10) 2. Note that $x^2(x-2) = x^3 - 2x^2$.
a) Find an equation for the line tangent to $y = x^3 - 2x^2$ when $x = 1$.
Answer $y' = 3x^2 - 4x$, so when $x = 1$, $y' = -1$. Also, when $x = 1$, $y = -1$. So an equation for the tangent line is $y - (-1) = (-1)(x-1)$.
b) Sketch the line found in a) and the curve $y = x^3 - 2x^2$ on the axes given below as well as you can. The units on the vertical and horizontal axes are different. **Answer** To the right.
c) For which x's are the tangent lines to the curve $y = x^3 - 2x^2$ horizontal?
Answer $3x^2 - 4x = 0$ when $x = 0$ and $x = \frac{4}{3}$.



(20) 3. Find the limit, which could be a specific real number or $+\infty$ or $-\infty$. In each case, briefly indicate your reasoning, based on algebra or properties of functions.
a) $\lim\limits_{x\to 2} \frac{\frac{1}{2}-\frac{1}{x}}{x-2}$ **Answer** $\frac{\frac{1}{2}-\frac{1}{x}}{x-2} = \frac{\frac{x-2}{2x}}{x-2} = \frac{1}{2x}$. As $x \to 2$, this $\to \frac{1}{4}$.
b) $\lim\limits_{x\to 4^+} \frac{4-x}{|4-x|}$. **Answer** If $x > 4$, then $0 > 4-x$ so $|4-x| = -(4-x)$. Therefore $\frac{4-x}{|4-x|} = \frac{4-x}{-(4-x)} = -1$. So the limit is -1.
c) $\lim\limits_{x\to 10^-} \frac{1}{100-x^2}$. **Answer** If $x < 10$ and close to 10, then $x^2 < 100$ and close to 100. So $100 - x^2$ is a small positive number, and $\frac{1}{100-x^2}$ will be a large positive number. The limit is $+\infty$.
d) $\lim\limits_{x\to\infty} \frac{1}{3e^x - 2e^{-x}}$ **Answer** As $x \to +\infty$, e^x grows unboundedly and e^{-x} decays to 0. Therefore $\frac{1}{3e^x-2e^{-x}} \to 0$.

(10) 4. Suppose $f(x) = x^2 - \frac{1}{x^3+12} + \sin(70x)$.
a) There is at least one number x between 0 and 2 for which $f(x) = 0$. Explain why this is true using complete English sentences together with appropriate references to results of this course.
Answer $f(0) = 0 - \frac{1}{12} + \sin 0 = -\frac{1}{12} < 0$. $f(2) = 4 - \frac{1}{20} + \sin(140)$. Since $\sin(140) \geq -1$, we see that $f(2)$ must be greater than $2\frac{19}{20}$, which is a positive number. Now f is a continuous function since rational functions in their domain and sine are continuous. Since $f(0) < 0 < f(2)$, the Intermediate Value Theorem implies that $f(x) = 0$ for at least one x in the open interval $0 < x < 2$.
b) If $x \geq 2$, $f(x)$ must be positive. Again, explain why this is true.
Answer If $x \geq 2$, $f(x)$ is at least $2^2 - \frac{1}{12} - 1$. This is because the largest negative number sine can be is -1, and when $x > 0$, $\frac{1}{x^3+12}$ is at most $\frac{1}{12}$ and x^2 is at least 2^2. But, as above, $2^2 - \frac{1}{12} - 1$ is positive, so $f(x)$ must be positive for $x > 2$.

(20) 5. Find $\frac{dy}{dx}$.
a) $y = 4x\left(5x^2-3\right)^7$ **Answer** $y' = 4\left(5x^2-3\right)^7 + 4x \cdot 7\left(5x^2-3\right)^6 (10x)$.
b) $y = \frac{\sin(4x)}{x^2+1}$ **Answer** $y' = \frac{\cos(4x)\cdot 4\cdot(x^2+1) - 2x\cdot\sin(4x)}{(x^2+1)^2}$.
c) $y = \sqrt{e^x + \cos(3x)}$ **Answer** $y' = \frac{1}{2}\left(e^x + \cos(3x)\right)^{-\frac{1}{2}} \cdot (e^x - \sin(3x)\cdot 3)$.
d) $x^3 + x^2y + 4y^3 = 6$ **Answer** $\frac{d}{dx}$ the equation and get $3x^2 + 2xy + x^2y' + 12y^2y' = 0$. Then solve for y' and get $y' = \frac{-3x^2-2xy}{x^2+12y^2}$.

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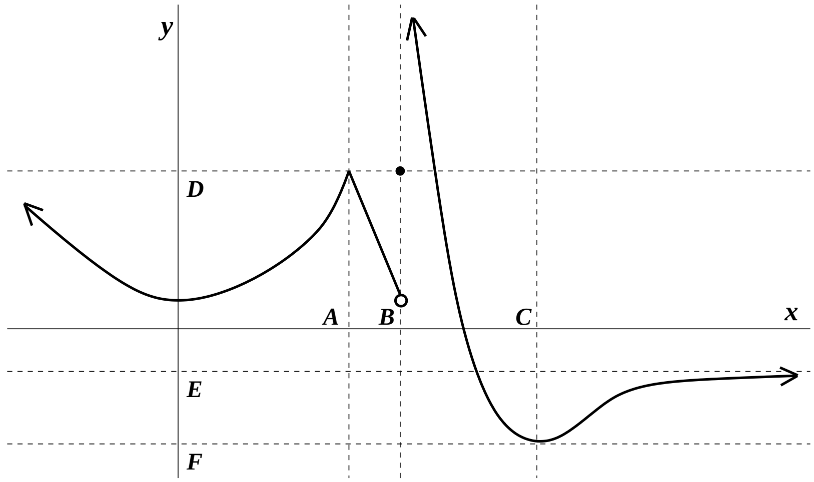
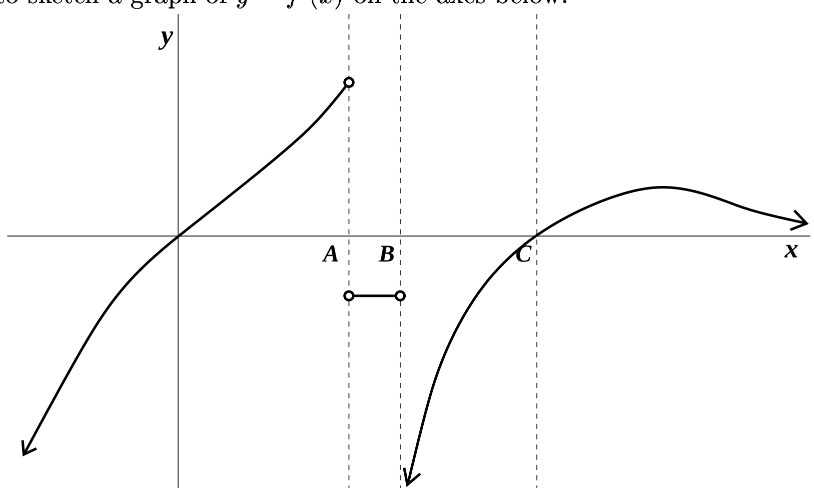
(18) 6. Here is a graph of $y = f(x)$.



a) Use this graph to sketch a graph of $y = f'(x)$ on the axes below.



b) Are there x's for which $f(x)$ is *not* continuous? If there are, list them. **Answer** Yes. $x = B$.
c) Are there x's for which $f(x)$ is *not* differentiable? If there are, list them. **Answer** Yes. $x = B$ and $x = A$.
d) Does $y = f(x)$ seem to have any horizontal asymptotes? If it does, write equations for any lines which seem to be horizontal asymptotes. **Answer** Yes. $y = E$.
e) Does $y = f(x)$ seem to have any vertical asymptotes? If it does, write equations for any lines which seem to be vertical asymptotes. equations for them. **Answer** Yes. $x = B$.

(10) 7. Find all lines tangent to $y = \frac{1}{x}$ which pass through the point $(-4, 2)$.
Answer Since $y' = -\frac{1}{x^2}$, and the slope of a line connecting $(x, y) = (x, \frac{1}{x})$ with $(-4, 2)$ is $\frac{\frac{1}{x}-2}{x-(-4)}$, we know that $-\frac{1}{x^2}$ should equal $\frac{\frac{1}{x}-2}{x-(-4)}$. If we cross-multiply, we get the equation $-(x-(-4)) = x^2\left(\frac{1}{x} - 2\right)$, and this becomes $-x-4 = x-2x^2$ so that we need to solve $2x^2 - 2x - 4 = 0$ or $x^2 - x - 2 = 0$. Amazingly (or not, since it *is* a problem on an exam!) the left-hand side factors into $(x+1)(x-2)$ so the roots of the equation are -1 and 2. When $x = -1$, the point on $y = \frac{1}{x}$ is $(-1, -1)$ and the slope is -1, so that the tangent line is $(y + 1) = (-1)(x + 1)$. When $x = 2$, the point on $y = \frac{1}{x}$ is $(2, \frac{1}{2})$ and the slope is $-\frac{1}{4}$, so that the tangent line is $(y - \frac{1}{2}) = (-\frac{1}{4})(x - 2)$. To the right is a picture of the two lines and the curve, a hyperbola, drawn by `Maple`.