Secret review problems for the final exam in Math 151:04–06, Fall 2003

The final exam will be cumulative, covering the whole course. Here are problems from previous Math 151 final exams. The rules for the final exam include the following:

- Calculators are *not* allowed.
- You may use a formula sheet supplied with the exam. A copy of this sheet is linked to the course web page. No other notes or texts will be allowed.

Solutions I hope students will send me solutions by *plain text* e-mail. If no solution is posted, please send e-mail with a solution for a problem whose letter begins your last name. Because almost 20% of the students in the class have last names beginning with **S** two problems have been given for that letter.

B Write the definition of derivative of a function as a limit and *use that definition* together with familiar properties of limits to find the derivative of $f(x) = \frac{1}{x+2}$.

C A farmer wants to enclose a rectangular field with total area of $10,000$ square feet. The rectangular field is to be further subdivided into three adjoining rectangular fields with two parallel interior fences (so the subfields share fences). What should the dimensions of all of the fences be in order to use the smallest possible length of fencing?
Remark Formal review problem #10 was too much like a problem that appeared on an old exam, so I made up this problem.

D Particle A moves along the positive x–axis and particle B moves along the positive y–axis.
a) If A is at the point $(a, 0)$ and B is at the point $(0, b)$, write an expression for the distance from A to B.
b) At a certain time, A is at the point $(7, 0)$ and moving *away from* the origin with a speed of 4 units/sec. At the same time B is at the point $(0, 5)$ and moving *towards* the origin at 2 units/sec. At what rate is the distance from A to B changing at that time? Are the particles moving towards each other or moving away from each other at that time?

E A particle is moving with the given data. Find a formula for the position, $s(t)$, of the particle at time t if $a(t) = \cos t + \sin t$, $s(0) = 0$, and $v(0) = 5$.
Here $v(t)$ is the particle's velocity at time t and $a(t)$ is its acceleration at time t.

F Suppose that Q is the function $Q(x) = 5 \arcsin(\ln x)$.
a) What are the domain and range of Q? Answers should *not* be numerical approximations, but should be written if needed in terms of constants studied in calculus such as π and e.
b) Suppose $y = Q(x)$. Write a formula for x in terms of y.

G Define the function V by $V(x) = \begin{cases} Cx^2 + (C^2 - C), & \text{if } x \geq 1 \\ 2x - C^2, & \text{if } x < 1 \end{cases}$ where C is a constant.
a) Find $\lim\limits_{x\to 1^-} V(x)$ and $\lim\limits_{x\to 1^+} V(x)$.
b) Which value or values of C make the function V continuous at $x = 1$? Explain in one sentence how the definition of continuity is used to determine your answers. Make a rough sketch of the curve $y = V(x)$ for *each* value of C which results in a continuous function.
c) Which value or values of C make the function V differentiable at $x = 1$? Explain in one sentence how the definition of differentiability is used to determine your answers.

H Suppose this is known about a twice differentiable function, H:

x	$H(x)$	$H'(x)$	$H''(x)$
1	2	0	2
2	3	6	5
3	7	3	-4
4	2	5	7

a) If w is a small number, write an approximation (the linearization of H at 2) for $H(2+w)$.

b) Is your answer in a) likely to be an underestimate of the true value of $H(2+w)$ when w is small, or an overestimate? Give a reason for your answer.

c) If $J(x) = \left(H(x)\right)^2$, compute $J(3)$, $J'(3)$, and $J''(3)$.

J Suppose $f(x) = \arctan(x^2)$.

a) What are $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ and why? Give the equations of any horizontal asymptotes.

b) Does the graph of $y = f(x)$ have any symmetries? Explain your answer algebraically.

c) Compute $f'(x)$. Find the roots of $f'(x) = 0$ algebraically. Where is $f'(x)$ positive and where is it negative? Explain what your results for f' (positive, negative, and zero) say about f itself.

d) Compute $f''(x)$. Find the roots of $f''(x) = 0$ algebraically. Where is $f''(x)$ positive and where is it negative? Explain what your results for f'' (positive, negative, and zero) say about f itself.

e) Does $f(x)$ have an absolute minimum on the interval $-\infty < x < +\infty$? Explain.
Does $f(x)$ have an absolute maximum on the interval $-\infty < x < +\infty$? Explain.

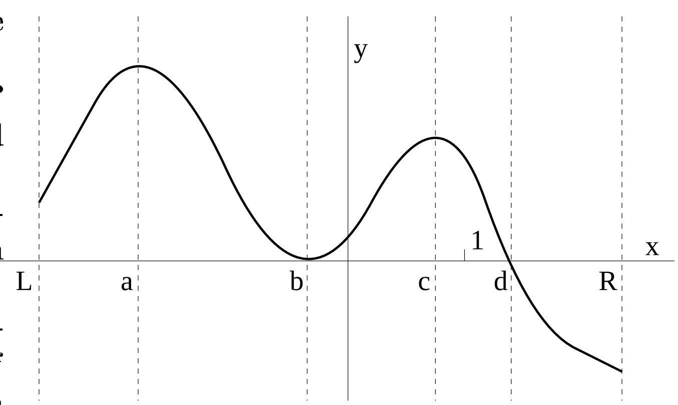
f) Sketch the graph of $y = f(x)$ as well as you can, illustrating the properties you have found.

K Here is a graph of the *derivative* of the function $f(x)$.

a) What are the critical numbers of f? Where does f have local extrema, and what kind of extrema are they?

b) What are the intervals in which f is increasing? What are the intervals in which f is decreasing?

c) Where are f's points of inflection located? What are the intervals in which f is concave up? What are the intervals in which T is concave down?



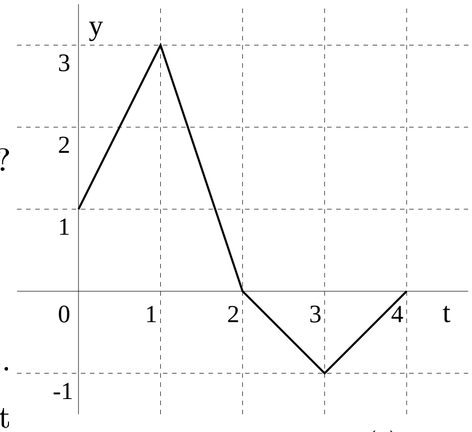
Graph of $y = f'(x)$, the derivative of $f(x)$

d) Suppose in addition that you know $f(0) = -1$. Use this information together with your answers to a), b) and c) to sketch a graph of $y = f(x)$ as well as you can. Use the letters on the graph of $f'(x)$ to label features on the graph of $f(x)$.

L In each case, find the limit. Numerical answers alone are *not* sufficient. What you write should show how you get your answers. Answers should *not* be numerical approximations but should be written if needed in terms of values of functions studied in calculus.

a) $\lim_{x\to+\infty}\left(\frac{x^3}{x^2-1} - \frac{x^3}{x^2+1}\right)$ b) $\lim_{x\to0}\frac{3^x-1}{5^x-1}$ c) $\lim_{x\to4}\frac{\sqrt{x}-2}{x+4}$

M The graph of f is shown. Let $g(x) = \int_0^x f(t)\,dt$.
a) Find $g(0)$ and $g(4)$.
b) Find $g'(1)$.
c) For what value of x does $g(x)$ have its maximum value?



The graph of $y = f(t)$

P Compute $\int_1^2 3x^2 - \frac{4}{x^2}\,dx$ using calculus.

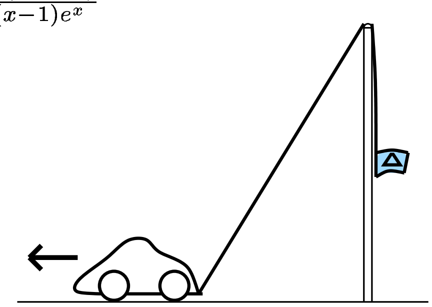
Q For the curve $y^3 + 5xy^2 + x^4 = -16$, find
a) $\frac{dy}{dx}$
b) An equation of the line tangent to the curve at $(-2, 2)$.

R Find the areas of the following regions. Give exact answers, as necessary in terms of well-known constants such as π, e, $\ln 3$, $\sqrt{2}$, etc.
a) The region above the x-axis, to the left of the line $x = 5$, and below the curve $y = e^x - 1$.
b) The region above the x-axis, between the lines $x = 1$ and $x = 2$, and below the curve $y = \frac{2}{x} + \sin(\pi x)$.

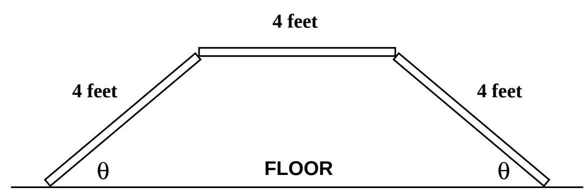
S₁ Suppose that $f(0) = 2$ and $f'(x) = \frac{1}{3+(\cos x)^2}$ for all x, $0 \le x \le \frac{\pi}{2}$. Using the Mean Value Theorem, you can conclude that $A \le f\left(\frac{\pi}{2}\right) \le B$ for certain numbers A and B, What are A and B?

S₂ Compute the following limits. Give exact answers, as necessary in terms of well-known constants such as π, e, $\ln 3$, $\sqrt{2}$, etc.
a) $\lim\limits_{x \to 0} \frac{\tan 17x}{\sin 13x}$ b) $\lim\limits_{x \to +\infty} \frac{\sqrt{4+5x^4}}{3+6x+7x^2}$ c) $\lim\limits_{x \to 1} \frac{\ln(2x^2-1)}{(x-1)e^x}$

T A flagpole is 40 feet high and stands on level ground. A flag is attached to a 120 foot rope passing through a pulley at the top of the flagpole. The other end of the rope is tied to a car at ground level. If the car is driving directly away from the flagpole at 3 ft/sec, how fast is the flag rising when the top of the flag is 20 feet off the ground?



U A child wants to build a tunnel using three equal boards, each 4 feet wide, one for the top and one for each side as shown. The two sides are to be tilted at equal angles θ to the floor. What is the maximum cross-sectional area A that can be achieved?



V Find the following:
a) $\int \frac{\cos 2x}{1+(\sin 2x)^2}dx$ b) $\int \left(\frac{4}{t^2} + \frac{1}{\sqrt{1-t^2}}\right) dt$

W Find the following: $\frac{d}{dx} \int_3^x \sqrt{t^7+1}\,dt$.

Y Find the following (give exact answers, not decimal approximations):
a) $\int_1^1 f(x)\,dx$ where $f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ x^3, & \text{if } x \ge 0 \end{cases}$. b) $\int_0^{\pi/4} (\sin x + \cos x)^2 dx$