1. Choose an appropriate starting guess and then use three iterations of Newton's method to find the smallest positive solution to

$$\frac{1}{1+x^2} = \tan x$$

How many positive solutions to this equation are there? Why? What would you guess would be true asymptotically about the consecutive positive solutions to this equation? (Pictures will help answer this question.)

2. `This is probably how your calculator computes multiplicative inverses.`†

a) Suppose $a = 1.2345$. The object of this problem is to study the Newton's method iteration for finding $\frac{1}{a}$.* Write the Newton's method iteration scheme for $f(x) = \frac{1}{x} - a$. That is, write a formula showing how an old guess, x, changes to a new and perhaps better guess. Please simplify the formula as much as possible so that it contains *no divisions.*

b) Suppose $x_0 = 1$ is the initial guess. How many iterations (repetitions) of Newton's method are needed to get $\frac{1}{a}$ to 10 digit accuracy? Note: $f(x) = 0$ when $x =$ ___.**

c) Now consider *any* starting point x_0 for Newton's method in this problem. Color x_0 **green** if the iteration of Newton's method converges to the only root of $f(x)$. Color x_0 **red** if the iteration of Newton's method does *not* converge to that root. Find an example of a red x_0 and a green x_0.

d) Continue your experimentation, supplemented with appropriate graphical and algebraic analysis. Find all red x_0's and all green x_0's. Discuss the solution as well as you can.***

3. The following statements are true facts:

$$(10{,}000{,}000{,}000)^{\left(\frac{1}{10{,}000{,}000{,}000}\right)} \approx 1.\,00000\,00\underline{23}\ \underline{02585}\ \underline{09}564$$

and

$$\ln 10 \approx \underline{2.30258}\ \underline{509}29$$

Explain the amazing coincidence of the digits.

Hint Approximate e^x when x is small.

4. Find the limits for the following indeterminate forms of the type "$\infty - \infty$".

a) $\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x}$ b) $\lim_{x\to 0} \frac{1}{x^2} - \frac{1}{x}$ c) $\lim_{x\to 0} \frac{1+x}{x} - \frac{1-x}{x}$

Note that three completely different types of behavior occur.

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† See the Intel reference in the course diary for October 20.

* To end the suspense, $\frac{1}{a}$ is .81004455245038477116 to twenty place accuracy.

** You fill in the blank.

*** Have fun with numbers, but a picture will probably serve you better after a while. And some algebra following the picture will be even better.