This is problem #6 from the first workshop.

6. Consider the parabola $y = x^2$. Flip it and move it right, to create a parabola opening "down" and intersecting the $x-$axis at $x = 1$ and $x = 2$. An equation for such a parabola is $y = -(x-1)(x-2)$. Look below: there is a straight line tangent to both parabolas. [The picture in the problem statement then appears.] Find an equation for the line.

What follows is one answer to the question.

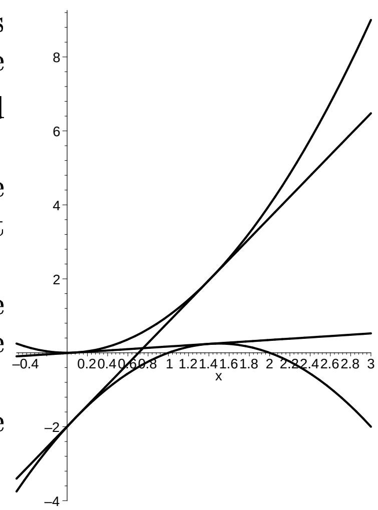
Suppose P is the point of tangency on $y = x^2$ and Q is the point of tangency on $y = -(x-1)(x-2)$. If P's first coordinate is s, then P has coordinates (s, s^2) and the line tangent to $y = x^2$ at P must have slope $2s$, the value of the derivative of x^2 when x is s. If Q's first coordinate is t, then Q has coordinates $(t, -(t-1)(t-2))$ and the line tangent to $y = -(x-1)(x-2) = -x^2 + 3x - 2$ at Q must have slope $-2t + 3$, the value of the derivative of $-x^2 + 3x - 2$ when x is t.

The line pictured in the problem is tangent to $y = x^2$ at P, so its slope must be $2s$. Since it is also tangent to $y = -(x-1)(x-2)$ at Q, its slope must be $-2t+3$. The points P and Q are distinct and lie on the line, so the slope must also be given by $\frac{s^2-(-(t-1)(t-2))}{s-t}$, which is obtained from the coordinates of P and Q. All three expressions must be equal. Since $2s = -2t + 3$, $s = -t + \frac{3}{2}$. Then $\frac{s^2-(-(t-1)(t-2))}{s-t}$ becomes $\frac{(-t+\frac{3}{2})^2-(-(t-1)(t-2))}{(-t+\frac{3}{2})-t}$ which is $\frac{2t^2-6t+\frac{17}{4}}{-2t+\frac{3}{2}}$. This must be the same as $-2t + 3$, so we get the following equation for t: $\frac{2t^2-6t+\frac{17}{4}}{-2t+\frac{3}{2}} = -2t + 3$. Multiply and get $2t^2 - 6t + \frac{17}{4} = (-2t + \frac{3}{2})(-2t + 3)$.

More multiplication and careful collection of terms results in a quadratic equation for t: $2t^2 - 3t + \frac{1}{4} = 0$. The roots are $\frac{3}{4} \pm \frac{\sqrt{7}}{4}$. I switched to approximate decimal computation, and got .08856 and 1.41143 as the values of t.

When $t = .08856$, the point P is $(1.41144, 1.99216)$. The line has slope 2.82288 and the equation of this doubly tangent line is $y = 2.82288x - 1.99212$.

When $t = 1.41143$, the point P is $(.8857, .00784)$. The line has slope .17714 and the equation of this doubly tangent line is $y = .17714x - .00784$.



`Maple`, a computer program, produced the picture of the parabolas and the **two** lines tangent to both parabolas.

Here are comments on the answer and the question.

1. The original picture in the problem statement may be deceptive. It did not show *two* lines doubly tangent to the parabolas. Many problems in applications, in school and (especially!) out of school, may have unexpected answers.
2. I wrote complete sentences. I did not present every algebraic or computational detail. These were summarized, but enough evidence was shown to be convincing.
3. For me, the key step in the explanation was writing distinct labels: I didn't use "x" for the first coordinate of each point of tangency, for the independent variable in several formulas, for the variable in each derivative, etc. Understanding and presenting the solution is nearly impossible without careful labels.