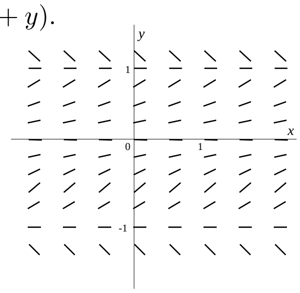
(8) 1. Find the solution of the differential equation $\frac{dy}{dx} = \frac{xy^3}{x^2+1}$ satisfying the initial condition $y(0) = 3$. In the answer express y explicitly as a function of x.

Answer This is a separable equation, so we get $\frac{dy}{y^3} = \frac{x\,dx}{x^2+1}$ which can be integrated to $-\frac{1}{2y^2} = (\frac{1}{2}\ln(x^2+1)) + C$. The initial condition $(0,3)$ gives the equation $-\frac{1}{18} = (\ln 1) + C$ so $-\frac{1}{2y^2} = (\frac{1}{2}\ln(x^2+1)) - \frac{1}{18}$ and, solving for y: $y = \frac{1}{\sqrt{\frac{1}{9} - \ln(x^2+1)}}$.

(12) 2. Below is part of the direction field for the differential equation $y' = y^2(1-y)(1+y)$.

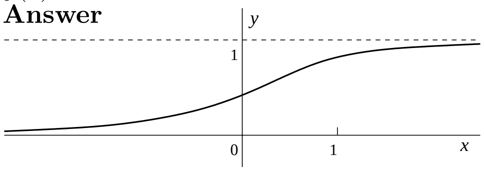


a) List all numbers k so that the constant function $f(x) = k$ is a solution of this differential equation (these are the *equilibrium solutions*).

Answer A constant function has derivative 0. Those $y = k$'s for which y' must always be 0 are the solutions of $y^2(1-y)(1+y) = 0$: k must be 0 or 1 or -1.

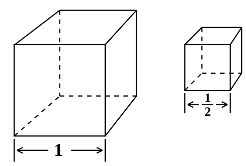
b) Sketch a typical solution curve $y = f(x)$ to this differential equation when $0 < y(0) < 1$.

Answer



The curve sketched should be located entirely within the strip $0 < y < 1$ and should be increasing with these asymptotic properties evident: $\lim_{x\to+\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 0$.

(12) 3. A sequence of cubes has edges made of thin wire. The largest cube has edge length 1 inch, and each successive cube has edge length half the size of the preceding one. The first two cubes are shown here.



a) What is the total length of wire needed to construct the edges on *all* of the cubes?

Answer Each cube has 12 edges. The n^{th} cube has edge length $\frac{1}{2^{n-1}}$, so the total length of wire needed is $\sum_{n=1}^{\infty} \frac{12}{2^{n-1}} = 24$ inches.

b) What is the total volume enclosed by *all* of the cubes?

Answer The edge length of the n^{th} cube is $\frac{1}{2^{n-1}}$ and since the volume of a cube is the cube of the edge length, the volume of that cube is $\left(\frac{1}{2^{n-1}}\right)^3 = \frac{1}{2^{3n-3}} = \frac{1}{8^{n-1}}$. The total volume enclosed is $\sum_{n=1}^{\infty} \frac{1}{8^{n-1}} = \frac{1}{1-\frac{1}{8}} = \frac{8}{7}$ cubic inches.

(10) 4. Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{3^n x^n}{\sqrt{n}}$. In addition, determine whether the series is absolutely or conditionally convergent at the boundary points of the interval of convergence.

Answer If $a_n = \frac{3^n x^n}{\sqrt{n}}$ then $a_{n+1} = \frac{3^{(n+1)} x^{(n+1)}}{\sqrt{n+1}}$ and $\left|\frac{a_n+1}{a_n}\right| = \frac{3^{(n+1)}|x|^{(n+1)}\sqrt{n}}{3^n|x|^n\sqrt{n+1}} = 3|x| \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$. The limit of this as $n \to \infty$ is $3|x|$ so, using the Ratio Test, the series converges when $|x| < \frac{1}{3}$ and diverges when $x > \frac{1}{3}$. When $x = \frac{1}{3}$ the series is $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, a $p-$series with $p = \frac{1}{2} < 1$ which diverges. When $x = -\frac{1}{3}$ the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, an alternating series which converges conditionally and not absolutely. The interval of convergence is $-\frac{1}{3} \le x < \frac{1}{3}$ and the radius of convergence is $\frac{1}{3}$

(10) 5. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{3n^2+5n+7}$ converges. Find a specific finite sum of rational numbers (quotients of integers) which is within .0001 of the sum of the infinite series. Be sure to explain why your error estimate is correct.

Hint Compare the "infinite tail" to something simpler, and analyze that.

Answer Since $\frac{1}{3n^2+5n+7} < \frac{1}{n^2}$, $\sum_{n=N+1}^{\infty} \frac{1}{3n^2+5n+7} < \sum_{n=N+1}^{\infty} \frac{1}{n^2}$. But $\sum_{n=N+1}^{\infty} \frac{1}{n^2} < \int_N^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big]_N^{\infty} = \frac{1}{N}$. If $N = 10{,}000$ the infinite tail $\sum_{n=N+1}^{\infty} \frac{1}{3n^2+5n+7} < .0001 = \frac{1}{10{,}000}$. The specific finite sum is $\sum_{n=1}^{10{,}000} \frac{1}{3n^2+5n+7}$.

(10) 6. a) Suppose the sequence $\{A_n\}$ is defined by $A_n = \dfrac{6 \cdot 4^n + 7n^3}{5 \cdot 4^n + 8n^2}$. What is $\lim_{n\to\infty} A_n$?

Answer $A_n = \frac{6\cdot 4^n+7n^3}{5\cdot 4^n+8n^2} = \frac{6+7\frac{n^3}{4^n}}{5+8\frac{n^2}{4^n}}$. But $\lim_{n\to\infty} \frac{n^3}{4^n} = 0$ and $\lim_{n\to\infty} \frac{n^2}{4^n} = 0$ (exponential growth is more rapid than polynomial growth), so $\lim_{n\to\infty} A_n = \frac{6}{5}$.

b) Suppose the sequence $\{B_n\}$ is defined by $B_n = \left(1 + \dfrac{3}{n}\right)^{5n}$. What is $\lim_{n\to\infty} B_n$? **Hint** ln and l'H.

Answer Take logs: $\ln(B_n) = \ln\left(\left(1+\frac{3}{n}\right)^{5n}\right) = 5n\ln\left(1+\frac{3}{n}\right) = \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{5n}}$. As $n \to \infty$, this is $\frac{0}{0}$ so try l'Hospital's rule: $\lim_{n\to\infty} \frac{\ln\left(1+\frac{3}{n}\right)}{\frac{1}{5n}} = \lim_{n\to\infty} \frac{\frac{d}{dn}\left(\ln\left(1+\frac{3}{n}\right)\right)}{\frac{d}{dn}\left(\frac{1}{5n}\right)}$ and we check if the latter limit exists. It is $\lim_{n\to\infty} \frac{\frac{1}{1+\frac{3}{n}}\cdot -\frac{3}{n^2}}{-\frac{1}{5n^2}} = \lim_{n\to\infty} \frac{15}{1+\frac{3}{n}} = 15$. Therefore $\lim_{n\to\infty} B_n = \lim_{n\to\infty} e^{\ln B_n} = e^{15}$.

(10) 7. Find a specific polynomial $p(x)$ so that $|p(x) - x^{1/4}| < .001$ for $16 \le x \le 17$. Be sure to explain why your error estimate is correct.

Answer Try a Taylor polynomial for $x^{1/4}$ centered at 16. Here $f(x) = x^{1/4}$ so $f(16) = 2$. $f'(x) = \frac{1}{4}x^{-3/4}$ and $f''(x) = -\frac{3}{16}x^{-7/4}$. Since $f'(16) = \frac{1}{4}16^{-3/4} = \frac{1}{32}$ we know that $T_1(x) = 2 + \frac{1}{32}(x-16)$. The error $|f(x) - T_1(x)|$ can be estimated by $M_2\frac{|x-16|^2}{2}$. Since $16 \le x \le 17$, $|x-16| \le 1$. M_2 is an overestimate of $\left|-\frac{3}{16}x^{-7/4}\right| = \frac{3}{16}x^{-7/4}$ on $[16, 17]$. M_2 can be estimated by $\frac{3}{16}16^{-7/4} = \frac{3}{2{,}048}$ because $x^{-7/4}$ is decreasing. So the error is at most $\frac{3}{2{,}048}\cdot\frac{1}{2} = \frac{3}{4{,}096} < \frac{1}{1{,}000} = .001$ and $p(x)$ can be $2 + \frac{1}{32}(x-16)$.

(12) 8. a) Use the Taylor series for the exponential function to write $\int_0^1 e^{-x^2}\,dx$ as an infinite series. Use summation notation in your answer.

Answer $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$ and $\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\Big]_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$.

b) Find a specific finite sum of rational numbers (quotients of integers) which is within .00001 of the true value of $\int_0^1 e^{-x^2}\,dx$. Be sure to explain why your error estimate is correct.

Answer The series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$ satisfies all the criteria for an alternating series, and therefore a partial sum $\sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)n!}$ will be within $\left|\frac{(-1)^{N+1}}{(2(N+1)+1)(N+1)!}\right| = \frac{1}{(2(N+1)+1)(N+1)!}$ (the first omitted term) of the true value. Since $.00001 = \frac{1}{100{,}000} > \frac{1}{(2(7+1)+1)\cdot 40{,}320}$, any $N \ge 7$ will serve. The desired specific approximation is $\sum_{n=0}^{7} \frac{(-1)^n}{(2n+1)n!}$. `Maple` reports the "true" value of the integral is .7468241330 and the value of the partial sum given is .<u>74682</u>28068 while the partial sum up to $n = 6$ is .<u>7468</u>360343 in decimal form.

(10) 9. Estimate the maximum error committed when $\cos x$ is replaced by $1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720}$ for x in the interval $[-2, 2]$. Be sure to explain why your error estimate is correct.

Answer The polynomial given is $T_7(x)$, the seventh-degree Taylor polynomial centered at 0 for $\cos x$. The error $|\cos x - T_7(x)|$ is overestimated by $M_8\frac{|x-a|^8}{8!}$. Here $a = 0$ and x is in the interval $[-2, 2]$. Therefore $|x - a| \le 2$. M_8 is an overestimate of the absolute value of the eighth derivative of cosine and so can't be larger than 1. Therefore $M_8\frac{|x-a|^8}{8!} \le \frac{256}{40{,}320}$.

(6) 10. a) What is the 300$^{\text{th}}$ Taylor polynomial for $f(x) = 3 - 11x^5$ centered at $a = 0$? Why?

Answer Since all the derivatives after the fifth are 0, the error term R_6 must always be 0. Therefore $T_{300}(x) = f(x)$.

b) Explain briefly why $g(x) = |x|$ has no Taylor series centered at $a = 0$.

Answer g is not differentiable at 0, and therefore $g'(0)$ does not exist.