

$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$	$f(x)$	$\int f(x) dx$
$x^r, r \neq -1$	$x^{r+1}/(r+1) + C$	$\sin x$	$-\cos x + C$	$1/\sqrt{a^2 - x^2}, a \neq 0$	$\arcsin(x/a) + C$
$x^{-1}$	$\ln x  + C$	$\cos x$	$\sin x + C$	$1/(a^2 + x^2), a \neq 0$	$(1/a) \arctan(x/a) + C$
$e^x$	$e^x + C$	$\sec^2 x$	$\tan x + C$	$\tan x$	$-\ln \cos x  + C$
$a^x, a \neq 1$	$a^x/(\ln a) + C$	$\sec x \tan x$	$\sec x + C$	$\sec x$	$\ln \sec x + \tan x  + C$

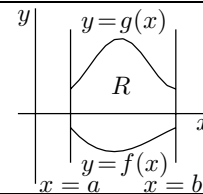
Areas and Volumes of Solids of Revolution, see figure at the right

$R$  is the region bounded by  $y = g(x), y = f(x), x = a, x = b$ , with  $g(x) \geq f(x)$  for  $x$  in  $[a, b]$ .

Area of  $R$ :  $\int_a^b (g(x) - f(x)) dx$

Volume of solid found by rotating  $R$  around the  $x$ -axis:  $\int_a^b \pi((g(x))^2 - (f(x))^2) dx, (f(x) \geq 0)$

Volume of solid found by rotating  $R$  around the  $y$ -axis:  $\int_a^b 2\pi x(g(x) - f(x)) dx, (a \geq 0)$



Integration by parts:  $\int u dv = uv - \int v du$ . Choose  $dv$  to be easy to integrate and so that  $\int v du$  is simpler than  $\int u dv$ .

Use for integrals of:  $x^m e^{ax}, x^m \sin ax, x^m \cos ax, x^m \ln x, x^m \arctan x, x^m \arcsin x, e^{ax} \sin bx, e^{ax} \cos bx, \sec^m x, m$  odd.

$\int \sin^m x \cos^n x dx$ : Reduce to a sum of integrals of the type:  $\int \sin^j x (\cos x dx)$  and  $\int \cos^k x (\sin x dx)$

$m$  odd: group  $\sin x$  with  $dx$ . Replace  $\sin^{m-1} x$  using  $\sin^2 x = 1 - \cos^2 x$ . Let  $u = \cos x, du = -\sin x dx$ . Expand.

$n$  odd: group  $\cos x$  with  $dx$ . Replace  $\cos^{n-1} x$  using  $\cos^2 x = 1 - \sin^2 x$ . Let  $u = \sin x, du = \cos x dx$ . Expand.

$m, n$  both even: Use  $\sin^2 x = (1 - \cos(2x))/2, \cos^2 x = (1 + \cos(2x))/2$ . Possibly repeat, or use an earlier case.

EX:  $\int \sin^5 x \cos^4 x dx = \int \sin^4 x \cos^4 x (\sin x dx) = \int (1 - \cos^2 x)^2 \cos^4 x (\sin x dx) = \int (1 - u^2)^2 u^4 (-du) = \int (-u^4 + 2u^6 - u^8) du = -(1/5)\cos^5 x + (2/7)\cos^7 x - (1/9)\cos^9 x + C$ . EX:  $\int \sin^2 x \cos^2 x dx = \int (1/4)(1 - \cos(2x))(1 + \cos(2x)) dx = (1/4) \int (1 - \cos^2(2x)) dx = (1/4) \int (1 - (1/2)(1 + \cos(4x))) dx = (1/8) \int (1 - \cos(4x)) dx = (1/8)x - (1/32)\sin(4x) + C$ .

$\int \sec^m x \tan^n x dx$ : Reduce to a sum of integrals of the type:  $\int \sec^j x (\sec x \tan x dx)$  and  $\int \tan^k x (\sec^2 x dx)$

$m$  even: group  $\sec^2 x$  with  $dx$ . Replace  $\sec^{m-2} x$  using  $\sec^2 x = 1 + \tan^2 x$ . Let  $u = \tan x, du = \sec^2 x dx$ . Expand.

$n$  odd: group  $\sec x \tan x$  with  $dx$ . Replace  $\tan^{n-1} x$  using  $\tan^2 x = \sec^2 x - 1$ . Let  $u = \sec x, du = \sec x \tan x dx$ .

$m$  odd,  $n$  even: Use  $\tan^2 x = \sec^2 x - 1$  to express as a sum of integrals of  $\sec^m x, m$  odd. Use integration by parts.

Integrals involving  $\sqrt{a^2 - x^2}$ : set  $x = a \sin \theta$ . Then  $dx = a \cos \theta d\theta$  and  $\sqrt{a^2 - x^2} = a \cos \theta$ .

Integrals involving  $\sqrt{a^2 + x^2}$ : set  $x = a \tan \theta$ . Then  $dx = a \sec^2 \theta d\theta$  and  $\sqrt{a^2 + x^2} = a \sec \theta$ .

Integrals involving  $\sqrt{x^2 - a^2}$ : set  $x = a \sec \theta$ . Then  $dx = a \sec \theta \tan \theta d\theta$  and  $\sqrt{x^2 - a^2} = a \tan \theta$ .

EX:  $\int x^5 (1 - x^2)^{3/2} dx = \int \sin^5 \theta \cos^3 \theta (\cos \theta d\theta) = \int \sin^5 \theta \cos^4 \theta d\theta = -(1/5)\cos^5 \theta + (2/7)\cos^7 \theta - (1/9)\cos^9 \theta + C = -(1/5)(1 - x^2)^{5/2} + (2/7)(1 - x^2)^{7/2} - (1/9)(1 - x^2)^{9/2} + C$

Integrals of rational functions:  $\int (f(x)/g(x)) dx$ , with  $f(x), g(x)$  polynomials.

(1) If  $\deg f(x) \geq \deg g(x)$ , divide  $f(x)$  into  $g(x)$  using long division of polynomials to get quotient  $q(x)$  and remainder  $r(x)$  with  $\deg r(x) < \deg g(x)$ . Then  $\int (f(x)/g(x)) dx = \int q(x) dx + \int (r(x)/g(x)) dx$ . Use next method on last integral.

(2) If  $\deg f(x) < \deg g(x)$ , use the method of partial fractions. First factor  $g(x)$  into a product of linear factors  $x + A$  and quadratic factors  $x^2 + Bx + C$  with no real roots, or  $g(x) = \underbrace{(x + A)^m \cdots (x + A)^m}_{\text{linear factors}} \underbrace{(x^2 + Bx + C)^n \cdots (x^2 + Bx + C)^n}_{\text{quadratic factors}}$ .

Then,  $f(x)/g(x)$  is a sum of terms:

$$\underbrace{\frac{D_1}{(x + A)} + \frac{D_2}{(x + A)^2} + \cdots + \frac{D_m}{(x + A)^m}}_{m \text{ terms for each linear factor } (x+A)^m} \text{ and } \underbrace{\frac{E_1 x + F_1}{(x^2 + Bx + C)} + \frac{E_2 x + F_2}{(x^2 + Bx + C)^2} + \cdots + \frac{E_n x + F_n}{(x^2 + Bx + C)^n}}_{n \text{ terms for each quadratic factor } (x^2+Bx+C)^n}$$

To find the constants  $D_i, E_j, F_k$ , multiply both sides by  $g(x)$ . Then multiply the terms out and equate the coefficients of the same powers of  $x$  on each side of the equation. Finally, solve the resulting system of linear equations.

EX:  $\frac{x^2+x+1}{x^2(x-1)(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{Dx+E}{x^2+1}$ . First multiply both sides by  $x^2(x-1)(x^2+1)$  to obtain:

$x^2 + x + 1 = Ax(x-1)(x^2+1) + B(x-1)(x^2+1) + Cx^2(x^2+1) + Dx^3(x-1) + Ex^2(x-1)$ , or  $x^2 + x + 1 = (A + C + D)x^4 + (-A + B - D + E)x^3 + (A - B + C - E)x^2 + (-A + B)x + (-B)$ . Thus,  $A + C + D = 0, -A + B - D + E = 0, A - B + C - E = 1, -A + B = 1, -B = 1$ . Solving,  $A = -2, B = -1, C = 3/2, D = 1/2, E = -1/2$ .

If  $g(x)$  is a product of distinct linear factors, there is a short cut. Multiply both sides by  $g(x)$  as before.

EX:  $\frac{x^2-2}{(x-2)x(x+1)} = \frac{A}{x-2} + \frac{B}{x} + \frac{C}{x+1}$ . Then:  $x^2 - 2 = Ax(x+1) + B(x-2)(x+1) + C(x-2)x$ . Let  $x$  equal each root of  $g(x)$ :  $x = 2, 0, -1$ . Then  $x = 2$  gives  $2 = 6A$ ;  $x = 0, -2 = -2B$ ;  $x = -1, -1 = 3C$ , so  $A = 1/3, B = 1, C = -1/3$ .

Suppose the integral  $\int_a^b f(x) dx$  is approximated by dividing  $[a, b]$  into  $n$  equal segments. If  $M_j$  is an overestimate for  $|f^{(j)}(x)|$  on  $[a, b]$ , the trapezoidal error is at most  $M_2(b-a)^3/12n^2$  and the Simpson's error is at most  $M_4(b-a)^5/180n^4$ .

Definitions:  $\int_a^\infty f(x) dx = \lim_{t \rightarrow \infty} \int_a^t f(x) dx$ ;  $\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx$ .  $\int_1^\infty \frac{dx}{x^a}$  converges if  $a > 1$ , diverges if  $a \leq 1$ .

Comparison: if  $0 \leq f(x) \leq g(x), I_f = \int_a^\infty f(x) dx, I_g = \int_a^\infty g(x) dx$  then: if  $I_g$  converges, so does  $I_f$ ; if  $I_f$  diverges, so does  $I_g$ .

Similar definitions and results for  $\int_a^b f(x) dx$  when  $f$  has bad behavior ( $\rightarrow +\infty$  or  $-\infty$ , say) at an endpoint. For example,  $\int_0^1 \frac{dx}{x^a}$  converges if  $a < 1$ , diverges if  $a \geq 1$ . Some limits: if  $a > 0, \lim_{x \rightarrow \infty} x^a = \infty, \lim_{x \rightarrow \infty} 1/x^a = 0$ . If  $a > 1, \lim_{x \rightarrow \infty} a^x = \infty, \lim_{x \rightarrow \infty} 1/a^x = 0$ .

Logs:  $\lim_{x \rightarrow \infty} \ln x = \infty; \lim_{x \rightarrow 0^+} \ln x = -\infty$ . And  $\lim_{x \rightarrow \pm\infty} \arctan x = \pm\pi/2$ .

L'Hospital: if  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}$  or  $\frac{\pm\infty}{\pm\infty}$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  (when the latter limit exists).