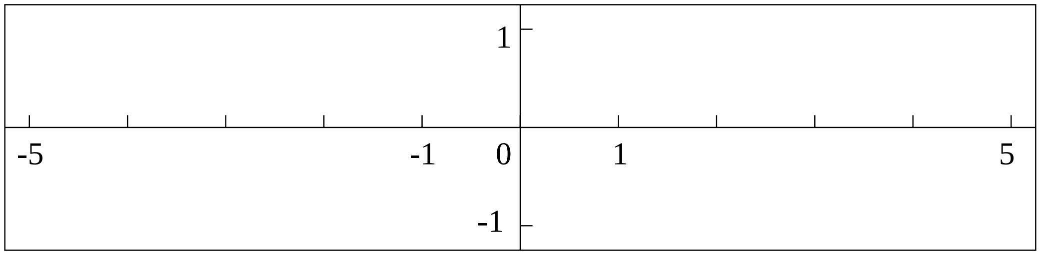
1. Sketch the graphs of the following functions in the indicated "window". Do these separately. Try not to use <u>any</u> calculator help!



a) $A(x) = \sin x$

b) $B(x) = \sin 2x$

c) $C(x) = 2\sin x$

d) $D(x) = (\sin x)^2$

e) $E(x) = \sin(x^2)$.

Remember: $\pi \approx 3.14$, so $\frac{\pi}{2} \approx 1.57$.

Pay attention to:

the <u>range</u> of the function, <u>symmetry and antisymmetry</u> with respect to the coordinate axes, and <u>local behavior near 0</u>.

2. a) Find positive numbers p and q so that

$$p \leq 6 - 2\cos(x^3) \leq q$$

is always true. Explain your answer briefly.

b) Find positive numbers r and s so that

$$r \leq 5\sin(3x^7 - 32) + 11 \leq s$$

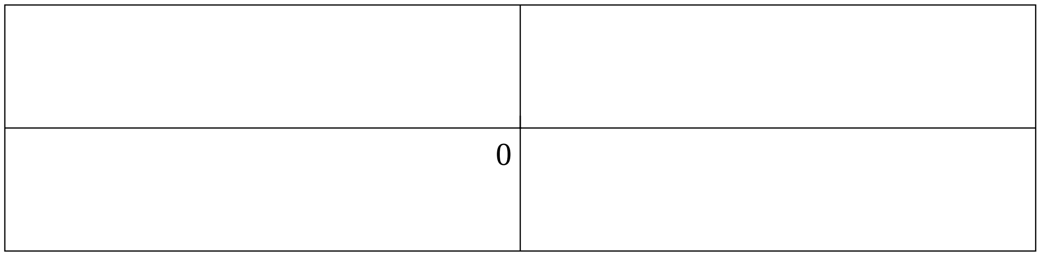
is always true. Explain your answer briefly.

c) Find positive numbers A and B so that

$$A \leq \frac{5\sin(3x^7 - 32) + 11}{6 - 2\cos(x^3)} \leq B$$

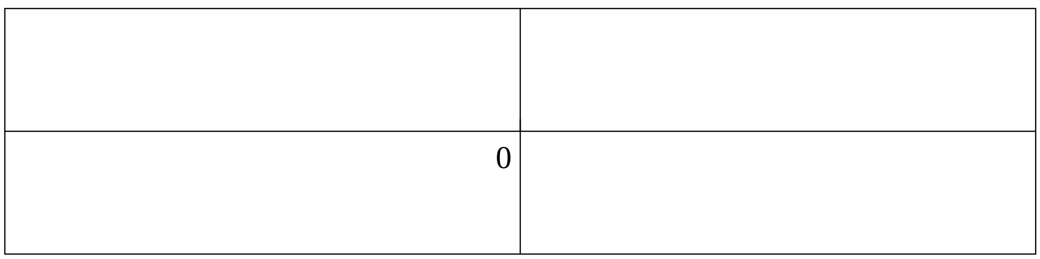
is always true. Explain your answer briefly.

3. a) Sketch a graph of $J(x) = \dfrac{1}{1+x^2}$.

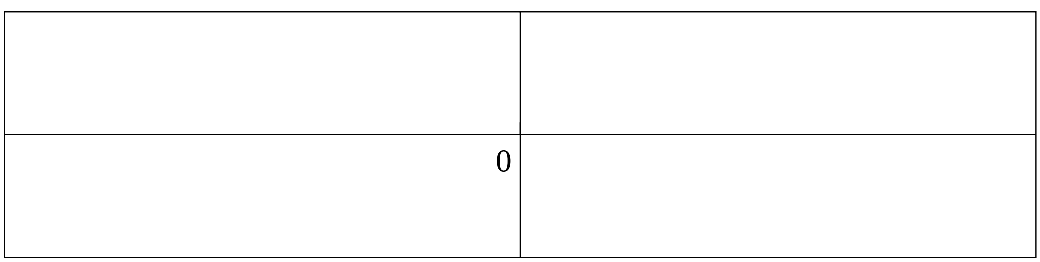


(<u>You</u> select the scale for the x-axis and the y-axis!)

b) Sketch a graph of $K(x) = \dfrac{12}{5+(x-3)^2}$.



c) Sketch a graph of $L(x) = \dfrac{8}{3+(x-1)^2} - \dfrac{7}{2+(x-15)^2}$.



4. a) Find positive numbers p and q so that

$$p \leq 100 - x^3 \leq q$$

is true for x in the interval $[1, 3]$.

Give a reason using calculus explaining your answer. Hint: derivative, {in|de}creasing.

b) Find positive numbers r and s so that

$$r \leq 4 + x^4 \leq s$$

is true for x in the interval $[1, 3]$.

Give a reason using calculus explaining your answer. Hint: derivative, {in|de}creasing.

c) Find positive numbers A and B so that

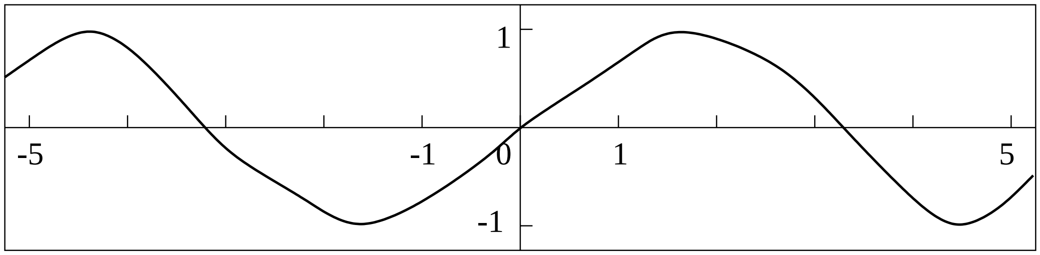
$$A \leq \frac{100 - x^3}{4 + x^4} \leq B$$

for true for x in the interval $[1, 3]$.

Why is your answer correct?

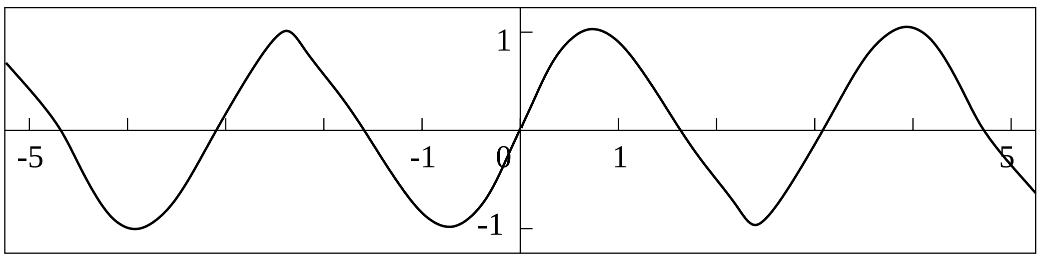
1. I DELIBERATELY TRIED TO DUPLICATE HOW I DREW THESE GRAPHS BY HAND. I DIDN'T USE ELECTRONIC ASSISTANCE. FOR BETTER PICTURES, PLEASE USE `Maple`.

a)



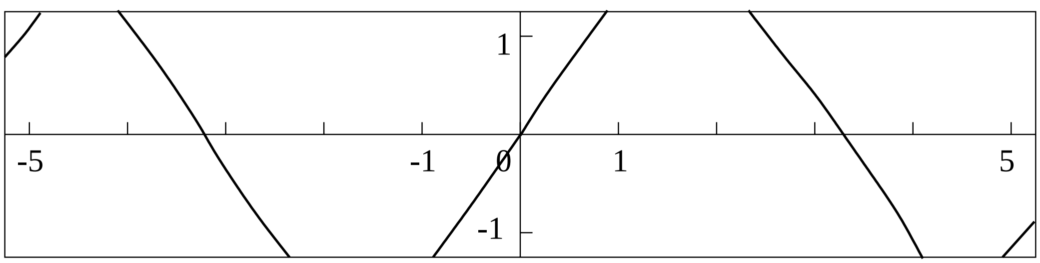
For x near 0, $\sin x \approx x$, the tangent line to $y = \sin x$ at $(0,0)$. Since $\sin(-x) = -\sin x$, the graph is antisymmetric ("odd"). Also, the range of $\sin x$ is all of $[-1,1]$.

b)



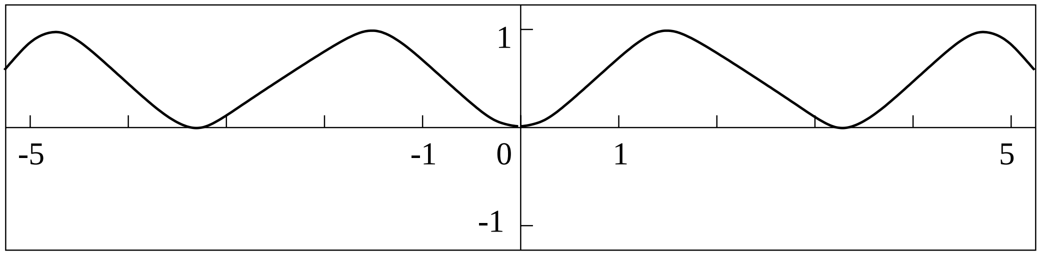
For x near 0, $\sin(2x) \approx 2x$. This graph is also antisymmetric. The range is $[-1,1]$. The period is π, half of the first graph's.

c)



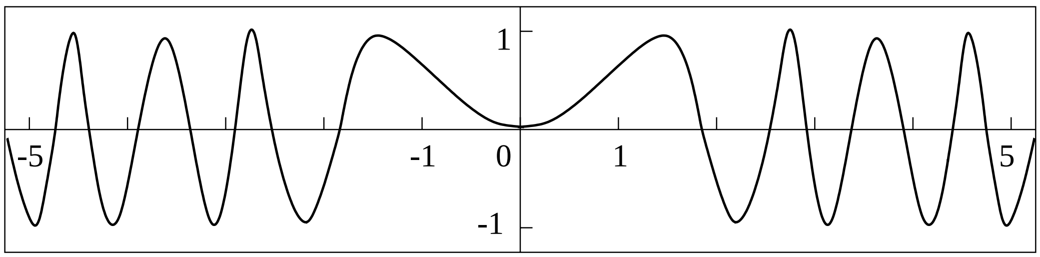
For x near 0, $2\sin x \approx 2x$. This graph is also antisymmetric. The range is $[-2,2]$. The period is 2π. OF COURSE, THIS IS A BIT OF A JOKE. THE "WINDOW" IS NOT ADEQUATE TO DISPLAY THE QUALITATIVE PROPERTIES OF THE GRAPH WELL.

d)



For x near 0, $(\sin x)^2 \approx x^2$. The graph is symmetric: $D(x) = D(-x)$. The range is $[0,1]$. The period is π, since $(\sin x)^2$ is $\frac{1}{2} - \frac{1}{2}\cos(2x)$. It seems remarkable to me that "squaring" a sine or cosine curve doubles its frequency.

e)



For x near 0, $\sin(x^2) \approx x^2$. This function is NOT periodic, but it is symmetric: $E(x) = E(-x)$. The range is $[-1,1]$. This graph was certainly the most difficult to draw by hand with no electronic help. To find the smallest positive x-intercepts, I needed to know when $\sin(x^2) = 0$. This occurs when $x^2 = \pi \approx 3.14$, so $x \approx \sqrt{3.14}$, about 1.8. Or $x^2 = 2\pi$, so $x \approx 2.5$. Or $x^2 = 3\pi$, so $x \approx 3.1$. Or $x^2 = 4\pi$, so $x \approx 3.5$. Or $x^2 = 5\pi$, so $x \approx 3.9$. Or $x^2 = 6\pi$, so $x \approx 4.3$. Or $x^2 = 7\pi$, so $x \approx 4.7$. That's the last positive x-intercept which appears in this window. Qualitatively, the important thing to note is that the function "wiggles" (oscillates is more polite!) faster as x increases. The situation is reflected for negative x.

Comment I wanted students to contrast b) and c) and also to contrast d) and e). The situations are somewhat irritating. The local behaviors near 0 of the functions in each pair are the same* but the "global" properties are rather different.

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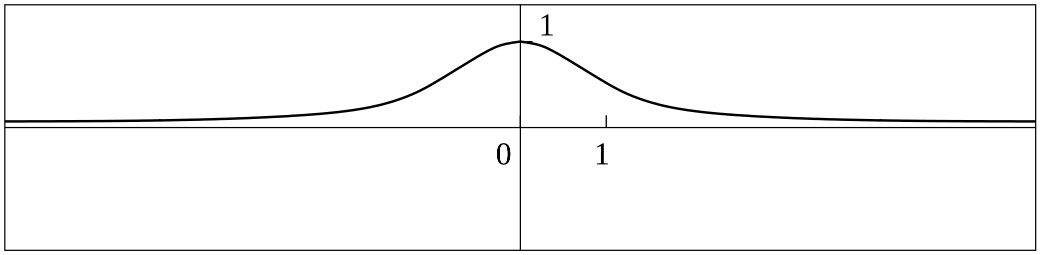
* In b) and c) the behavior is like $2x$, and in d) and e), like x^2.

2. a) Values of cosine are always between -1 and $+1$, so $6 - 2\cos(x^3)$ will be between 4 and 8, which are acceptable values of p and q.

b) Values of sine are always between -1 and $+1$, so $5\sin(3x^7 - 32) + 11$ will be between 6 and 16, which are acceptable values of r and s.

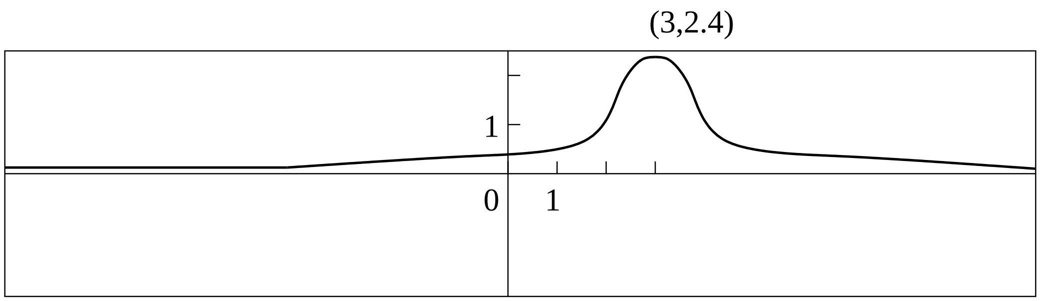
c) A quotient of positive functions is always $\leq \dfrac{\text{Overestimate of the top}}{\text{Underestimate of the bottom}}$ and is $\geq \dfrac{\text{Underestimate of the top}}{\text{Overestimate of the bottom}}$, so you can take $A = \dfrac{r}{q} = \dfrac{6}{8}$ and $A = \dfrac{s}{p} = \dfrac{16}{4}$.

3. a) The graph of $\frac{1}{1+x^2}$ is <u>unimodal</u>, a word commonly used in statistics and various applications. It has a single peak:

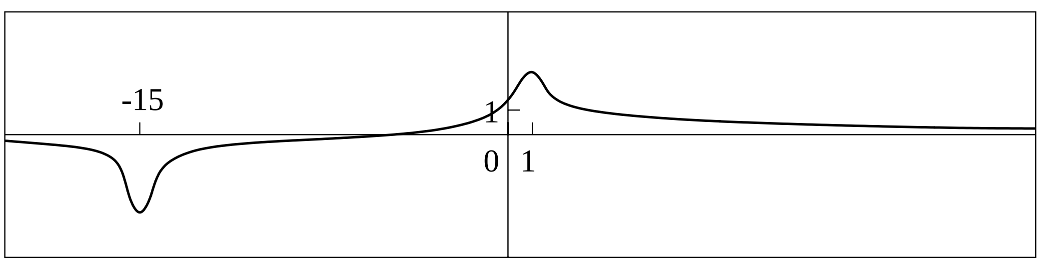


The bump is not sharp (like the peak of a roof), since the function is differentiable. It must be a smooth maximum.

b) The peak is at 3 and has height $\frac{12}{5}$.



c) The difference of two peaks (the graph is approximate!).



The bumps are far enough from each other so that they *probably* don't interact very much.

4. a) Suppose $f(x) = 100 - x^3$. Then $f'(x) = -3x^2$, always negative, so f is decreasing. Therefore if x is in the interval $[1, 3]$,

$$\underset{\substack{\text{Left-hand}\\\text{endpoint}}}{f(1)} \geq f(x) \geq \underset{\substack{\text{Right-hand}\\\text{endpoint}}}{f(3)}$$

Take $p = f(3) = 100 - 3^3 = 100 - 27 = 73$, and $q = f(1) = 100 - 1^3 = 100 - 1 = 99$.

a) Suppose $g(x) = 4 + x^4$. Then $g'(x) = 4x^3$. On $[1, 3]$, this is always positive, so g is decreasing. Therefore if x is in the interval $[1, 3]$,

$$\underset{\substack{\text{Left-hand}\\\text{endpoint}}}{g(1)} \leq g(x) \leq \underset{\substack{\text{Right-hand}\\\text{endpoint}}}{g(3)}$$

Take $r = g(1) = 4 + 1^4 = 4 + 1 = 5$, and $s = g(3) = 4 + 3^4 = 4 + 81 = 85$.

c) As in the answer to 2c),

$$\frac{\text{Min Top}}{\text{Max Bottom}} \leq \text{Quotient} \leq \frac{\text{Max Top}}{\text{Min Bottom}}.$$

So take $A = \frac{73}{85}$ and $B = \frac{99}{5}$.