1. Suppose that $F(x) = 10{,}000{,}000\ln(x)$ and $G(x) = 100x^3$ and $H(x) = e^{.00001x}$.

 A positive number x will be colored RED if the largest of the three numbers, $\{F(x), G(x), H(x)\}$ is $F(x)$. x will be colored GREEN if $G(x)$ is the largest. x will be colored BLUE if $H(x)$ is the largest.

a) **Do the following with explicit computation.** Try to do it without a calculator.

What color is 10? Note: $\ln 10 \approx 2.303$.
Find a specific number x which is RED.
Find a specific number x which is GREEN.
Find a specific number x which is BLUE.

b) **Do the following with an appropriate method for analyzing limits.**
What color are most numbers?

Hint To compare RED and BLUE you should examine the asymptotic behavior of $F(x)$ versus $G(x)$. So you should look at $\frac{F(x)}{G(x)}$ as $x \to \infty$, which you can investigate with l'Hospital's rule. The winning "color" in this pair should compete against $H(x)$.

2. Consider the following functions:

$$\ln(x^5) \quad x^8 \quad e^{(x^2)} \quad \sqrt{x} \quad \ln(\ln(x)) \quad (\ln(x))^2 \quad (e^x)^2$$

Make a list, in order, of which is bigger as $x \to \infty$.

Let me be precise: I will write $f \prec g$ if $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. For example, using this notation, $5x + 7{,}000 \prec x^2$ because $\displaystyle\lim_{x\to\infty} \frac{5x + 7{,}000}{x^2} = 0$.

Here's the task: write the functions above from "smallest" to "largest" with $\prec$ notation. This is not easy and you should verify your assertions by computing the appropriate limits.

1. a) $F(10) \approx 10{,}000{,}000 \cdot 2.303 = 23{,}030{,}000$ and $G(10) = 10^2 \cdot 10^3 = 10^5 = 100{,}000$ and $H(10) = e^{.00001 \cdot 10} = e^{.0001} < e$. Certainly $F(10)$ is largest, so 10 is RED.

For a GREEN number, try $x = 1{,}000 = 10^3$. Then $G(10^3) = 100 \cdot (10^3)^3 = 10^2 \cdot 10^9 = 10^{11}$, while $F(10^3) = 10{,}000{,}000 \ln(10^3) = 30{,}000{,}000 \ln(10) \approx 6.909 \cdot 10^7$ and $H(10^3) = e^{.00001 \cdot 10^3} = e^{.01}$ and this is still less than e.

Let's try $x = 10^{100}$ for a BLUE number. $F(10^{100}) = 10{,}000{,}000 \ln(10^{100}) \approx 2,303{,}000{,}000$, and $G(10^{100}) = 10^{302}$ while $H(10^{100}) = e^{.00001 \cdot 10^{100}} = e^{\left(10^{95}\right)}$. How big in terms of powers of 10 is such a number? Certainly $e^3 > 10$ (actually, e^3 is about 20). Since $10^{95} > 3 \cdot 10^{94}$ and $e^{3 \cdot 10^{94}} = \left(e^3\right)^{10^{94}}$, $e^{\left(10^{95}\right)} > 10^{\left(10^{94}\right)}$, which is 1 followed by 10^{94} zeros. That's a much, much bigger number than the others.

b) Most numbers are BLUE. To verify this, use l'Hospital's rule.

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} = \lim_{x\to\infty} \frac{10{,}000{,}000 \ln(x)}{100x^3} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{10^7 \cdot \frac{1}{x}}{300x^2} = 0$$

which certainly shows that most numbers can't be RED.

Now for $\lim\limits_{x\to\infty} \dfrac{G(x)}{H(x)}$. This is:

$$\lim_{x\to\infty} \frac{100x^3}{e^{.00001x}} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{300x^2}{.00001e^{.00001x}} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{600x}{(.00001)^2 e^{.00001x}} \overset{\text{l'H}}{=} \lim_{x\to\infty} \frac{600}{(.00001)^3 e^{.00001x}} = 0$$

Therefore eventually $H(x)$ is (much!) bigger than both $G(x)$ and $F(x)$, so most numbers are BLUE.

2. Here is the list, ordered using $\prec$:

$$\ln(\ln(x)) \prec \ln(x^5) \prec (\ln(x))^2 \prec \sqrt{x} \prec x^8 \prec (e^x)^2 \prec e^{(x^2)}$$

To verify this, you must show that all of the following limits are 0:

$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{\ln(x^5)} \text{ and } \lim_{x\to\infty} \frac{\ln(x^5)}{(\ln(x))^2} \text{ and } \lim_{x\to\infty} \frac{(\ln(x))^2}{\sqrt{x}} \text{ and } \lim_{x\to\infty} \frac{\sqrt{x}}{x^8} \text{ and }$$
$$\lim_{x\to\infty} \frac{x^8}{(e^x)^2} \text{ and } \lim_{x\to\infty} \frac{(e^x)^2}{e^{(x^2)}}.$$

These limits can all be computed with l'Hospital's rule. If you haven't done this yet, **<u>do it now</u>** before you go on!!

3. a) What is a geometric series with first term a and ratio r?

b) Under what conditions does such a series converge? What is its sum when it converges?

c) Which of the following could be the initial segments of geometric series?

i) $1 - .01 + .0001 - .000001 + .00000001 + \ldots$
ii) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$
iii) $Q + T^7Q^2 + T^{14}Q^3 + T^{21}Q^4 + \ldots$
iv) $Q + T^7Q^2 + T^{14}Q^4 + T^{21}Q^9 + \ldots$

Explain your conclusions. In each case where you believe the series is a geometric series, tell what "a" and "r" seem to be.

4. a) In a certain geometric series, the fourth term is -1 and the seventh term is $\frac{1}{8}$. Find the sum of the series.

b) In a certain geometric series, the first term is 11 and the eleventh term is 16 times the fifteenth term. Find the sum of the series.

3. a) A geometric series is a series each term of which equals the preceding term multiplied by a fixed constant. If the first term is a and the multiplier is r, the series begins $a + ar + ar^2 + ar^3 + \ldots$. Using summation notation, this is $\sum_{n=0}^{\infty} ar^n$.

b) The series converges if $|r| < 1$, and its sum is $\frac{a}{1-r}$. If $a = 0$ the series converges and its sum is 0. The series does not converge for other values of a and r.

c) i) We look at ratios of successive terms:

$$\frac{-.01}{1} = -.01 \text{ and } \frac{.0001}{-.01} = -.01 \text{ and } \frac{-.000001}{.0001} = -.01 \text{ and } \frac{.00000001}{-.000001} = -.01$$

which seems correct if the series is a geometric series. The first term is 1 and the ratio is $-.01$ which certainly has absolute value less than 1, so the series converges.

ii) Here we see:

$$\frac{\frac{1}{3}}{1} = \frac{1}{3} \text{ and } \frac{\frac{1}{5}}{\frac{1}{3}} = \frac{3}{5}$$

and since these differ it is already impossible for the series to be a geometric series.

iii) We compute:

$$\frac{T^7Q^2}{Q} = T^7Q \text{ and } \frac{T^{14}Q^3}{T^7Q^2} = T^7Q \text{ and } \frac{T^{21}Q^4}{T^{14}Q^3} = T^7Q$$

and since these are all the same, this could indeed be an initial segment of a geometric series with $a = Q$ and $r = T^7Q$.

iv) Compute again:

$$\frac{T^7Q^2}{Q} = T^7Q \text{ and } \frac{T^{14}Q^4}{T^7Q^2} = T^7Q^3 \text{ and } \frac{T^{21}Q^9}{T^{14}Q^4} = T^7Q^5$$

and because these are different, the series given is not a geometric series. (Although to be logically complete, if either T or Q is 0, this *is* a geometric series!)

4. a) The "fourth term" is ar^3, and the "seventh term" is ar^6. The counting method I'm using has a as first term. So we know that $ar^3 = -1$ and $ar^6 = \frac{1}{8}$. We can solve for r:

$$\frac{ar^6}{ar^3} = \frac{\frac{1}{8}}{-1} \text{ so } r^3 = -\frac{1}{8} \text{ so } r = -\frac{1}{2}\,.$$

Now for a: since $ar^3 = -1$, we know $a\left(-\frac{1}{2}\right)^3 = -1$ which gives $-\frac{1}{8}a = -1$ so that a must be 8. The sum is $\frac{a}{1-r}$ and here this is $\frac{8}{1-\left(-\frac{1}{2}\right)} = \frac{16}{3}$.

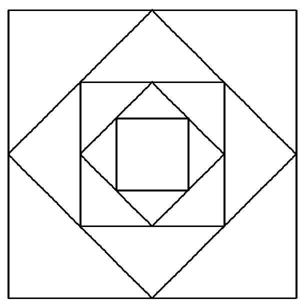
b) We know $a = 11$. The eleventh term is ar^{10} and the fifteenth term is ar^{14}. Then "the eleventh term is 16 times the fifteenth term" becomes the equation $ar^{10} = 16ar^{14}$ so that $1 = 16r^4$. Since $16 = 2^4$, r must be $\frac{1}{2}$. Here $\frac{a}{1-r}$ becomes $\frac{11}{1-\frac{1}{2}} = 22$.

5. a) Two students are sharing a loaf of bread. Student Alpha eats half of the loaf, then student Beta eats half of what's left, then Alpha eats half of what's left, and so on. How much of the loaf will each student eat?

b) Two students are sharing a loaf of bread. Student Alpha, now hungrier and more ferocious, eats two-thirds of the loaf, then student Beta eats half of what's left, then Alpha eats two-thirds of what's left, then Beta eats half of what's left, and so on. How much of the loaf will each student eat?

c) Now let's start with three students: Alpha, Beta, and Gamma. They decide to share a loaf of bread. Alpha eats half of the loaf, passes what's left on to Beta who eats half, and then on to Gamma who eats half, and then back to Alpha who eats half, and so on. How much of the loaf will each student eat?

6. An infinite sequence of squares is drawn (the first five are shown), with the midpoints of the sides of one being the vertices of the next. The outermost square has sides which are 1 unit long. What is the <u>sum</u> of the <u>perimeters</u> of all of the squares?



5. a) Alpha eats $\frac{1}{2}$ and then Beta eats $\frac{1}{4}$ and passes on $\frac{1}{4}$. So Alpha's second "bite" is half of that, or $\frac{1}{8}$, and $\frac{1}{8}$ is passed to Beta, who eats half of that, which is $\frac{1}{16}$. Let's list these:

Alpha eats: $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \dots$

Beta eats: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \dots$

The total bread Alpha eats is a geometric series whose first term is $\frac{1}{2}$ with ratio $\frac{1}{4}$. The total is therefore $\frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{4}} = \frac{2}{3}$.

The total bread Beta eats is a geometric series whose first term is $\frac{1}{4}$ with ratio $\frac{1}{4}$. The total is therefore $\frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$.

We can see that the sum of the bread eaten is 1, one loaf! Also, there are other logically correct ways to analyze this problem. The method presented here merely takes advantage of our knowledge of sums of geometric series.

b) A similar analysis gives Alpha's portion now as $\frac{12}{15} = \frac{4}{5}$ and Beta's portion as $\frac{1}{5}$.

c) A similar analysis gives Alpha's portion now as $\frac{4}{7}$, Beta's portion as $\frac{2}{7}$, and Gamma's portion as $\frac{1}{7}$.

6. Look at the process of going from one square to its successor. Two halves of each side of a square get replaced by a hypotenuse; that is, $\frac{1}{2}S + \frac{1}{2}S \longrightarrow \sqrt{2}\frac{1}{2}S$. This occurs four times. So $4S \longrightarrow 2\sqrt{2}S$ as we go from the perimeter of one square to the next. We start with the biggest square which has perimeter 4 ($S = 1$) and see that we must sum a geometric series whose first term is 4 and whose ratio between successive terms is $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$. Thus the sum of the perimeters is $\frac{a}{1-r} = \frac{4}{1-\left(\frac{1}{\sqrt{2}}\right)}$.