Arc length of $y=f(x)$ from $x=a$ to $x=b$ is $\int_a^b \sqrt{1+(dy/dx)^2}\,dx = \int_a^b \sqrt{1+(f'(x))^2}\,dx$.

Differential equations–separation of variables. Factor one side of the equation so that $dy/dx = f(x)g(y)$. Then $dy/g(y) = f(x)dx$. Integrate, not forgetting the constant C of integration. Use the initial condition $(x,y)=(x_0,y_0)$ to find C.
EX: For $dy/dx = x+xy$, with $y=2$, when $x=0$, then $dy/dx = x(1+y)$, or $dy/(1+y) = x\,dx$. Integrating, $\ln(1+y) = x^2/2+C$. So $1+y = e^{x^2/2+C}$, or $y = Ae^{x^2/2}-1$, with $A=e^C$. Set $x=0$, $y=2$, $2=A-1$ and $y=3e^{x^2/2}-1$.

Exponential population growth: $P=P_0e^{kt}$, where P_0 is the initial population and k is a constant.

$\lim\limits_{n\to\infty}\dfrac{n^a}{b^n}=0$, if $b>1$; $\lim\limits_{n\to\infty}\dfrac{a^n}{n!}=0$; f,g poly., $\lim\limits_{n\to\infty}\dfrac{f(n)}{g(n)} = \lim\limits_{n\to\infty}\dfrac{an^j+\text{lower terms}}{bn^k+\text{lower terms}} = \begin{cases} a/b, & \text{if } j=k; \\ 0, & \text{if } j<k;\ \pm\infty, \text{ if } j>k. \end{cases}$

$\lim\limits_{n\to\infty} c^{1/n}=0$, if $c>0$; $\lim\limits_{n\to\infty} n^{1/n}=0$; $\lim\limits_{n\to\infty}(1+(1/n))^n = e$. Use L'Hospital's Rule to verify these results.

Geometric series $\sum_{n=0}^\infty x^n = 1/(1-x)$, if $|x|<1$. $\sum_{n=1}^\infty 1/n^p$ converges if $p>1$, diverges if $p\le 1$.

Divergence Test: The series $\sum a_n$ diverges if either $\lim_{n\to\infty} a_n$ does not exist or exists and is not 0.

Integral Test: Suppose that f is a positive, continuous, decreasing function. Let $a_n=f(n)$ for each positive integer n. Then the integral $\int_1^\infty f(x)\,dx$ and the series $\sum_{n=1}^\infty a_n$ converge or diverge together. Use for $\sum 1/n^p$, $\sum 1/(n(\ln n)^p)$.
EX: For $\sum_{n=2}^\infty \frac{1}{n\sqrt{\ln n}}$, find $\int_2^\infty \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^\infty \frac{du}{\sqrt{u}} = 2\sqrt{u}\Big|_2^\infty = \infty$, using $u=\ln x$, $du=\frac{dx}{x}$. The series diverges.

Comparison Test: Assume $0\le a_n\le b_n$. If $\sum b_n$ converges, so does $\sum a_n$. If $\sum a_n$ diverges, so does $\sum b_n$.
EX: Note that $\frac{1}{n^2\ln n}\le\frac{1}{n^2}$, if $n\ge 3$, as $\ln n\ge 1$, if $n\ge 3$. Since $\sum\frac{1}{n^2}$ converges, so does $\sum\frac{1}{n^2\ln n}$.

Limit Comparison: If $a_n,b_n>0$, $\lim_{n\to\infty}(a_n/b_n)=L$, with $L\ne 0,\infty$, $\sum a_n$ and $\sum b_n$ both converge or diverge.
EX: Compare $\sum\frac{\sqrt{n}}{n+1}$ to $\sum\frac{1}{\sqrt{n}}$, since $\left(\frac{\sqrt{n}}{n+1}\right)\Big/\left(\frac{1}{\sqrt{n}}\right) = \frac{n}{n+1}\to 1$, as $n\to\infty$. Since $\sum\frac{1}{\sqrt{n}}$ diverges, both diverge.

Alternating Series: If $\sum_{n=0}^\infty(-1)^na_n$ is a series with $a_n\ge 0$, $a_{n+1}\le a_n$ for all n and $\lim_{n\to\infty}a_n=0$, the series converges.

Absolute Convergence: The series $\sum a_n$ converges absolutely if $\sum|a_n|$ converges. Then $\sum a_n$ converges also.
Conditional Convergence: The series $\sum a_n$ converges conditionally if $\sum a_n$ converges but $\sum|a_n|$ diverges.

Ratio Test: Suppose $\lim_{n\to\infty}|a_{n+1}/a_n|=L$. If $L<1$, $\sum a_n$ converges. If $L>1$, $\sum a_n$ diverges. If $L=1$, the test fails. Often useful for series involving a^n or $n!$. Root Test: Similar but with $\lim_{n\to\infty}|a_n|^{1/n}=L$. These tests can be used to find radius of convergence for power series.
EX: For $\sum\frac{3^n}{n!}$, $\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}}{(n+1)!}\cdot\frac{n!}{3^n} = \frac{3}{n+1}\to 0$, as $n\to\infty$. As $L=0<1$, the series converges absolutely.

Series	Test
$\sum\frac{1}{n(lnn)^4}$	Integral
$\sum\frac{n^2}{(n+1)2^n}$	Ratio
$\sum\frac{(-1)^nn}{n^2+1}$	Alter. Ser.

Series	Test
$\sum\frac{(n!)^2}{(2n)!}$	Ratio
$\sum\frac{n^2+1}{n^2+n}$	Divergence
$\sum\frac{n^2+n}{n^4+1}$	Lim. Comp. to $\sum\frac{1}{n^2}$

Series	Test
$\sum\frac{1}{(\ln n)2^n}$	Comp: $\sum\frac{1}{(\ln n)2^n}\le\sum\frac{1}{2^n}$
$\sum\frac{\ln n}{n}$	Comp: $\sum\frac{1}{n}\le\sum\frac{\ln n}{n}$
$\sum\frac{\sin n}{n^2}$	Comp: $\sum\frac{\lvert\sin n\rvert}{n^2}\le\sum\frac{1}{n^2}$

Estimating Sums with the Integral Test: The error in approximating $\sum_{n=1}^\infty a_n$ with $\sum_{n=1}^N a_n$ is at most $\int_N^\infty f(x)dx$.
EX: How large should N be to insure that $\sum_{n=1}^N ne^{-n^2}$ is within $1/10^{10}$ of $\sum_{n=1}^\infty ne^{-n^2}$? We need $\int_N^\infty xe^{-x^2}\,dx\le 1/10^{10}$. Integrating, $e^{-N^2}/2\le 1/10^{10}$, or $10^{10}/2\le e^{N^2}$, or $10\ln 10+\ln 2\le N^2$, or $\sqrt{10\ln 10+\ln 2}\le N$, N an integer.
With Alternating Series: The error in approximating $\sum_{n=0}^\infty(-1)^na_n$ with $\sum_{n=0}^N(-1)^na_n$ has absolute value at most a_{n+1}.

Radius of convergence: For the power series, $\sum c_n(x-a)^n$, there is a number $R\ge 0$ so that if $|x-a|<R$, the series converges absolutely and if $|x-a|>R$, the series diverges. If $|x-a|=R$, the series may converge or diverge. Find R with the ratio test. The interval of convergence I is the set of points where the series converges. I contains the interval $(a-R,a+R)$ and may or may not contain the boundary points $a-R$ and $a+R$.
EX: For $\sum\frac{nx^{2n}}{3^n}$, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)|x|^{2n+2}}{3^{n+1}}\cdot\frac{3^n}{n|x|^{2n}} = \left(\frac{n+1}{n}\right)\cdot\frac{|x|^2}{3}\to\frac{|x|^2}{3}$, as $n\to\infty$. The series converges absolutely if $|x|^2/3<1$, or $|x|^2<3$, or $|x|<\sqrt{3}$. If $|x|=\sqrt{3}$, the series is $\sum n$ and diverges. Then, $R=\sqrt{3}$ and $I=(-\sqrt{3},\sqrt{3})$.

EX: Use power series integration to find the power series of $\arctan x$ at $a=0$. Setting $x=-t^2$ in the geometric series: $\frac{1}{1+t^2}=\sum_{n=0}^\infty(-1)^nt^{2n}$ and $\arctan x = \int_0^x\frac{dt}{1+t^2} = \int_0^x\sum_{n=0}^\infty(-1)^nt^{2n}\,dt = \sum_{n=0}^\infty(-1)^n\int_0^x t^{2n}\,dt = \sum_{n=0}^\infty\frac{(-1)^nx^{2n+1}}{2n+1}$.

Taylor Polynomials: The n-th Taylor poly. of f is $T_n(x) = f(a)+f'(a)(x-a)+f''(a)(x-a)^2/2!+\cdots+f^n(a)(x-a)^n/n!$, assuming f has n derivatives at a. Also $f(x)=T_n(x)+R_n(x)$, $|R_n(x)|\le M|x-a|^{n+1}/(n+1)!$, with M an upper bound of $|f^{n+1}(t)|$ with t in the interval with endpoints a and x. EX: Find $T_2(x)$ of $f(x)=(1+x)^{1/2}$ at $a=0$. Estimate $|R_2(x)|$ with x in $[0,.1]$. Here, $f(x)=(1+x)^{1/2}$, $f'(x)=(1/2)(1+x)^{-1/2}$, $f''(x)=(-1/4)(1+x)^{-3/2}$, $f'''(x)=(3/8)(1+x)^{-5/2}$. Then, $f(0)=1$, $f'(0)=1/2$, $f''(0)=-1/4$. Thus, $T_2(x)=1+x/2-x^2/8$. Also, $|R_2(x)|\le M|x|^3/3!$, with M an upper bound of $|f'''(t)|=(3/8)(1+t)^{-5/2}$, with t in $[0,.1]$. Since $1+t$ is increasing, $(3/8)(1+t)^{-5/2}$ is decreasing on $[0,.1]$ and its maximum value occurs at $t=0$ and is $3/8$. Thus, $M=3/8$, $|x|\le .1$, and $|R_2(x)|\le(3/8)(.1)^3/6=(.1)^3/16$.

$e^x=\sum_{n=0}^\infty x^n/n!$; $\sin x=\sum_{n=0}^\infty(-1)^nx^{2n+1}/(2n+1)!$; $\cos x=\sum_{n=0}^\infty(-1)^nx^{2n}/(2n)!$; $\ln(1+x)=\sum_{n=0}^\infty(-1)^nx^{n+1}/(n+1)$

Parametric curves: If $x=f(t), y=g(t)$, then the slope of the tangent line is given by $\frac{dy}{dx}=\left(\frac{dy}{dt}\right)/\left(\frac{dx}{dt}\right)=\frac{g'(t)}{f'(t)}$. The arc length from $t=a$ to $t=b$ is $\int_a^b\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b\sqrt{(f'(t))^2+(g'(t))^2}\,dt$. Polar Coordinates: $x=r\cos\theta$, $y=r\sin\theta$. $r=\sqrt{x^2+y^2}$, $\theta=\arctan(y/x)$. The area bounded by $y=f(\theta)$, $\theta=a$, $\theta=b$ is $\int_a^b\frac{1}{2}r^2\,d\theta=\int_a^b\frac{1}{2}(f(\theta))^2\,d\theta$.

DRAFT