1. Decide if the series converges.
   - If it does, find a partial sum within $.000001 = \frac{1}{1{,}000{,}000}$ of the true sum.
   - If the series does not converge, find (if this is possible) a specific partial sum which is greater than 1,000,000.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,(1+\ln n)^2}$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4^n+\sqrt{n^5+8}}$

AFTER DOING THIS QUESTION PLEASE GET PAGE **A1** TO CHECK YOUR ANSWER.

2. Determine which of the following statements are always true. If a statement is false, give an example to show that it fails.

a) If a series doesn't converge, then its n^{th} term does not approach 0 as n approaches ∞ .

b) If the n^{th} term of a series does not approach 0 as n approaches ∞, then the series diverges.

c) If all of the partial sums of a series are less than some constant L, then the series converges.

d) If a series converges, then there is a constant L such that all of its partial sums are less than L.

AFTER DOING THIS QUESTION PLEASE GET PAGE **A2** TO CHECK YOUR ANSWER.

3. a) How many terms in the following series do you need to add in order to give an answer within $\frac{1}{10}$ of the true value?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\,.$$

b) We will show later that

$$\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

If you add up the same number of terms of this series as you did in part a), how close will you be to the true value?

4. Does $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(203n^5 - 47n^2 + 29.888)}{n^{1.008}}$ converge?

AFTER DOING THESE TWO QUESTIONS PLEASE GET PAGE **A3** TO CHECK YOUR ANSWER.

5. Use the Ratio Test to decide for which x's the following series converge. In each case there will be exceptional values of x for which the Ratio Test gives no information. Please use other methods to decide if the resulting series converge.

a) $\displaystyle\sum_{n=1}^{\infty} 7n^5x^n$ b) $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^5}x^n$

c) $\displaystyle\sum_{n=1}^{\infty} 12n^{1/5}x^n$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^{1/5}}x^n$

AFTER DOING THIS QUESTION PLEASE GET PAGE **A4** TO CHECK YOUR ANSWER.

6. For which values of x does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3x^{4n}}{(3n)!}$ converge? (Be careful: $3n!$ and $(3n)!$ and $3(n+1)!$ and $(3(n+1))!$ are all distinctly different!)

AFTER DOING THIS QUESTION PLEASE GET PAGE **A5** TO CHECK YOUR ANSWER.

6. Here $a_n = \dfrac{(n!)^3 x^{4n}}{(3n)!}$ and $a_{n+1} = \dfrac{((n+1)!)^3 x^{4(n+1)}}{(3(n+1))!}$ so that

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{((n+1)!)^3 x^{4(n+1)}}{(3(n+1))!}}{\frac{(n!)^3 x^{4n}}{(3n)!}} = \frac{((n+1)!)^3 x^{4(n+1)}(3n)!}{(3(n+1))!(n!)^3 x^{4n}}.$$

This takes some untangling. The powers of x are easy. There is x^{4n+4} on top and x^{4n} on the bottom, resulting in x^4 on the top.
Now the factorials. On top we have $((n+1)!)^3 = \big((n+1)n!\big)^3 = (n+1)^3(n!)^3$ which will be balanced against the $(n!)^3$ on the bottom. On the bottom we have
$(3(n{+}1))! = (3n{+}3)! = (3n{+}3)(3n{+}2)! = (3n{+}3)(3n{+}2)(3n{+}1)! = (3n{+}3)(3n{+}2)(3n{+}1)(3n)!$
which will be balanced against the $(3n)!$ on the top.
So after all this the quotient becomes $\dfrac{(n+1)^3 x^4}{(3n+3)(3n+2)(3n+1)}$ and we need to decide what happens to this as $n \to \infty$. Certainly $\frac{n+1}{3n+3} \to \frac{1}{3}$ and $\frac{n+1}{3n+2} \to \frac{1}{3}$ and $\frac{n+1}{3n+1} \to \frac{1}{3}$, all limits which can be checked with l'Hospital's rule.
So the quotient $\to \frac{x^4}{27}$ as $n \to \infty$. The Ratio Test then tells us that the series converges absolutely if $|x| < 27^{1/4}$ and diverges if $|x| > 27^{1/4}$.

5. a) Here $a_n = 7n^5x^n$ and $a_{n+1} = 7(n+1)^5x^{n+1}$, so that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{7(n+1)^5x^{n+1}}{7n^5x^n}\right| = \left(1+\frac{1}{n}\right)^5 |x|$$

and as $n \to \infty$ this $\to |x|$. So the Ratio Test says that this series converges if $|x| < 1$ and diverges if $|x| > 1$.
What happens if $|x| = 1$? There are two cases. When $x = 1$, the n^{th} term is $7n^5$ and certainly this does *not* $\to 0$ as $n \to \infty$. Similarly, when $x = -1$, the n^{th} term is $(-1)^n 7n^5$ and this also doesn't get small. So the series does not converge when $x = 1$ or $x = -1$.

b) Here a_n is $\frac{3}{n^5}x^n$ so the quotient is:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{3}{(n+1)^5}x^{n+1}}{\frac{3}{n^5}x^n}\right| = \left|\frac{3n^5x^{n+1}}{3(n+1)^5x^n}\right| = \left(\frac{n}{n+1}\right)^5 |x|$$

and this $\to |x|$ as $n \to \infty$.
Therefore the Ratio Test says that the series converges if $|x| < 1$ and diverges if $|x| > 1$. If $|x| = 1$ the Ratio Test gives no information and we must look separately at these cases.
When $x = 1$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^5}1^n$, and this is just 3 times a p-series with $p = 5 > 1$ so it converges.
When $x = -1$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n^5}(-1)^n$, and the absolute value of the terms of this series are the preceding series, which converges. So this series converges absolutely and therefore must converge.

c) The algebra resembles the preceding problem, and the quotient is

$$\left|\frac{a_{n+1}}{a_n}\right| = |x| \left(\frac{n+1}{n}\right)^{1/5}$$

which $\to |x|$ as $n \to \infty$. Therefore the series converges if $|x| < 1$ and diverges if $|x| > 1$ by the Ratio Test.
We must examine the situation where $|x| = 1$ more carefully. When $x = 1$, the n term of the series is $12n^{1/5}$ and this certainly does not $\to 0$ as $n \to \infty$, so the series doesn't converge. And similarly the series does not converge when $n = -1$.

d) Here again the series converges if $|x| < 1$ and diverges if $|x| > 1$. When $x = 1$, we get $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^{1/5}}$ which diverges since it is a multiple of a p-series with $p = \frac{1}{5} < 1$. When $x = -1$, though, the series becomes $\displaystyle\sum_{n=1}^{\infty} \frac{8}{n^{1/5}}(-1)^n$, a series which satisfies all of the hypotheses of the Alternating Series Test and which therefore converges.

3. a) This series satisfies the Alternating Series Test. A partial sum is always within (the absolute value of) the first omitted term. Here the first omitted term is $\frac{1}{N+1}$ in absolute value (omitted from $S_N = \displaystyle\sum_{n=1}^{N} \frac{(-1)^n}{n}$). This will be less than $\frac{1}{10}$ when N is at least 10.

b) So how close is $\displaystyle\sum_{n=0}^{9} \frac{(-1)^n}{n!}$ to the true value of the infinite series? Note that I wrote the first **10** terms of this series, even though the index of summation only goes up to 9.The first omitted term has absolute value $\frac{1}{10!}$ which is $\frac{1}{3{,}628{,}800}$. This is of course much smaller than $\frac{1}{10}$. This series converges faster than the series in a).

4. Yes. Look at absolute values of each term and estimate the top:

$$\frac{|\sin(203n^5 - 47n^2 + 29.888)|}{n^{1.008}} \le \frac{1}{n^{1.008}}.$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1.008}}$ is a p-series with $p = 1.008 > 1$, it converges. Therefore the original series in the problem converges *absolutely*, and since absolute convergence implies convergence, it also must converge.

2. a) False. The implication is complicated, but it says that a divergent series *must* have n^{th} term $\not\to 0$. But the harmonic series diverges, and its n^{th} term is $\frac{1}{n}$ which does $\to 0$.

b) This is true. The n^{th} term is the difference between successive partial sums. These successive partial sums each approach the sum of the whole series, so their difference *must* $\to 0$.

c) This is false. Consider $\sum_{n=1}^{\infty} a_n$ with $a_n = (-1)^n$. The first partial sum is -1 (in fact, all odd partial sums are -1!). All of the even partial sums are 0. So all of the partial sums are certainly less than 203, for example. So if $L = 203$, the conclusion is false but the hypotheses are correct.

d) Yes, this *is* true. All the partial sums must cluster around the sum of the whole series (which is the limit of the partial sums), so there are only a finite number of them more than 1 unit away from the limit. So the maximum of the partial sums is the maximum of 1+the sum of the series and the maximum of that finite number of partial sums far away from the limit.

1. a) Try the integral test. We need to compute $\int_1^\infty \frac{1}{x(1+\ln x)^2}dx$. To find an antiderivative, try $u = \ln x$ so $du = \frac{1}{x}dx$ and $\int \frac{1}{x(1+\ln x)^2}dx = \int \frac{1}{(1+u)^2}du = -\frac{1}{1+u} + C = -\frac{1}{1+\ln x} + C$. Then $\int_1^A \frac{1}{x(1+\ln x)^2}dx = -\frac{1}{1+\ln x}\Big]_{x=1}^{x=A} = -\frac{1}{1+\ln A} - \left(-\frac{1}{1+\ln 1}\right) = -\frac{1}{1+\ln A} + 1$. As $A \to \infty$, this $\to 1$ since logs get big when their arguments get big.
So the integral converges and therefore the sum converges.
We split the infinite sum:

$$\sum_{n=1}^{\infty} \frac{1}{n\,(1+\ln n)^2} = \sum_{n=1}^{N} \frac{1}{n\,(1+\ln n)^2} + \sum_{n=N+1}^{\infty} \frac{1}{n\,(1+\ln n)^2}$$

where the first sum is S_N, the partial sum, and the second is T_N, the "infinite tail". By putting the boxes "under the curve", it can be seen* that $T_N \le \int_N^\infty \frac{1}{x(1+\ln x)^2}dx = \frac{1}{1+\ln N}$. If we get $\frac{1}{1+\ln N} < \frac{1}{1{,}000{,}000}$ then the infinite tail will also be less than .000001. So we need $1{,}000{,}000 < 1+\ln N$ and this is the same as $e^{999{,}999} < N$: an **enormous** number. `Maple` reports that this is about $10^{434{,}285}$.
This series converges, but very very **very** slowly.

b) This series also converges. Split this series up as we did in a) into a partial sum and an infinite tail. Then the first chore is to throw out some irrelevant things:

$$\frac{1}{4^n + \sqrt{n^5+8}} \le \frac{1}{4^n}$$

so that the infinite tail can be overestimated by $\sum_{n=N+1}^{\infty} \frac{1}{4^n}$. Here recognizing the series will save a great deal of work. This is a *geometric series* with first term $a = \frac{1}{4^{N+1}}$ and ratio $r = \frac{1}{4}$. So the sum is

$$\frac{a}{1-r} = \frac{\frac{1}{4^{N+1}}}{1-\frac{1}{4}} = \frac{\frac{1}{4^{N+1}}}{\frac{3}{4}} = \frac{1}{3 \cdot 4^N}$$

and we need to make this less than $\frac{1}{1{,}000{,}000}$. Our goal is therefore $4^N > \frac{1{,}000{,}000}{3} \approx 333{,}333$. Here are some powers of 4: $4 = 4^1, 16 = 4^2, 64 = 4^3, 256 = 4^4, 1024 = 4^5, 4096 = 4^6$, $16{,}384 = 4^7, 65{,}536 = 4^8, 262{,}144 = 4^9$, and finally $4^{10} = 1{,}048{,}576$. So $N = 10$ is enough.

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* Ask if you don't "see"!