Sneaky questions!

Special advice: The questions on this page need almost *no* computation but do need some thought.

1. a) Can you find a power series centered at 0 which converges at $x = 3$ and diverges at $x = -2$? Give an example of one if you can, or give a reason why if you can't.

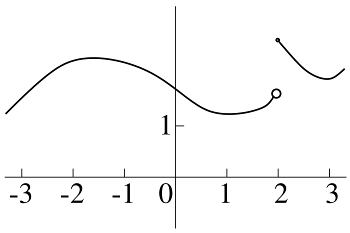
b) Can you find a power series centered at 0 which converges only at $x = 0$? Give an example of one if you can, or give a reason why if you can't.

c) Can you find a power series centered at 0 which converges only for x's in the interval $-3 \le x < 3$ (read carefully!)? Give an example of one if you can, or give a reason why if you can't.

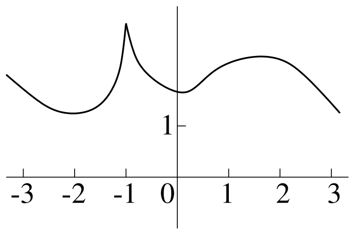
2. The graphs of two functions, $V(x)$ and $W(x)$, are shown below. Answer the following as well as you can by considering these graphs.

a) What is the largest possible radius of convergence of a power series representing $V(x)$ centered at 0? Give a reason for your answer.

b) What is the largest possible radius of convergence of a power series representing $W(x)$ centered at 0? Give a reason for your answer.



Graph of $y = V(x)$



Graph of $y = W(x)$

3. Yesterday in class we considered the function $S(x)$ whose power series was $0x^0 + 1x^1 + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 \ldots$. All of the terms (except x^0) have positive integer coefficients. So when $x > 0$ each of the terms is positive. We also saw that $S(x)$ had an alternative description: $S(x) = \dfrac{-x}{x^2 + x - 1}$. Clearly if $x = 10$, say, the top of this expression is negative and the bottom is positive, so that the quotient is negative.

Question Is it positive or negative? Explain the disagreement.

After doing these questions please get page **A1** to check your answers.

1. a) No such power series can be found. A series which converges at $x = 3$ must have radius of convergence at least equal to 3, and so it must converge absolutely (and converge!) for all x's with $-3 < x < 3$. Since -2 is in that interval, the series must converge at $x = -2$.

b) Yes, there are many such series. If the series is $\sum a_n x^n$, all that's needed is to specify a sequence of coefficients $\{a_n\}$ which tend to ∞ very quickly. For example, if $a_n = n^n$, the series will converge only for $x = 0$ (see this using the Root Test). Or, if $a_n = n!$, the series will converge also only for $x = 0$ (see this using the Ratio Test).

c) The series $\sum_{n=1}^{\infty} x^n$ (the geometric series) converges for $-1 < x < 1$. Can we "adjust" it to converge for $-3 < x < 3$? The simplest adjustments are $\sum_{n=1}^{\infty} 3x^n$ and $\sum_{n=0}^{\infty} \frac{1}{3}x^n$ which don't change the radius of convergence. But try $\sum_{n=1}^{\infty} 3^n x^n$ and $\sum_{n=1}^{\infty} \frac{1}{3^n}x^n$. The first has radius of convergence $\frac{1}{3}$ and the second has radius of convergence equal to 3. Now consider $\sum_{n=1}^{\infty} \frac{1}{3^n}x^n$ more closely. When $x = 3$ this series is $\sum_{n=1}^{\infty} \frac{1}{3^n}3^n = \sum_{n=1}^{\infty} 1$ and this diverges. When $x = -3$ this series is $\sum_{n=1}^{\infty} \frac{1}{3^n}(-3)^n = \sum_{n=1}^{\infty}(-1)^n$ which also diverges. But there is some hope, since there's sign alternation. We can modify the series we're considering with coefficients that come from a series which converges conditionally and not absolutely such as $\frac{1}{\sqrt{n}}$. The series $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}x^n$ still has radius of convergence equal to 3 (the Ratio Test shows that easily). When $x = 3$ the series is $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}3^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ and this diverges (p-series with $p = \frac{1}{2} < 1$). When $x = -3$ the series is $\sum_{n=1}^{\infty} \frac{1}{3^n\sqrt{n}}(-3)^n = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ and this series converges (the Alternating Series Test).

2. a) The sum of a power series is *continuous* inside its radius of convergence. But V's graph shown here has a discontinuity (a jump) at $x = 2$. So a power series for $V(x)$ can't converge at $x = 2$. The largest radius of convergence consistent with this is $R = 2$.

b) The sum of a power series is *differentiable* inside its radius of convergence. But the graph shown here for $W(x)$ has a sharp corner at $x = -1$. So a power series for $W(x)$ can't converge at $x = -1$. The largest radius of convergence consistent with this is $R = 1$.

3. Certainly $S(10)$ is negative. Certainly each term in the power series is positive. But when $x = 10$ the series and the function have nothing to do with each other. The series converges when $|x| < \frac{\sqrt{5}-1}{2} \approx .618$. And when $0 < x < .618$, $S(x)$ is positive, as are all the terms of the power series are!

4. Suppose $f(x)$ is the sum of a power series whose initial terms are $3-7x+5x^2-19x^4+\ldots$.
a) What is $f(0)$? What is $f^{(4)}(0)$?

b) What are the first three non-zero terms of a power series for $\left(f(x)\right)^2$?

c) What are the first three non-zero terms of a power series for $f(x^3)$?

Suppose now that $g(x)$ is a function which is the sum of a power series whose initial terms are $8+2x-3x^2-7x^3+\ldots$.

d) What are the first three non-zero terms of a power series for $f(x)+g(x)$ (the sum)?

e) What are the first three non-zero terms of a power series for $f(x)\cdot g(x)$ (the product)?

AFTER DOING THIS QUESTION PLEASE GET PAGE **A2** TO CHECK YOUR ANSWERS.

5. a) Find a power series for $\dfrac{1}{1-x}$. What is its radius of convergence?

ADVT. $\dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} ar^n$ if $|r|<1$. ADVT.

b) Find a power series for $\dfrac{1}{1+x}$. What is its radius of convergence?

c) Find a power series for $\dfrac{1}{1-x^3}$. What is its radius of convergence?

d) Find a power series for $\dfrac{1}{1+x^3}$. What is its radius of convergence?

e) Find a power series for $\displaystyle\int \frac{1}{1+x^3}\,dx$. What is its radius of convergence?

f) Use your answer to e) and the error estimate for Alternating Series to find an approximation to $\displaystyle\int_0^{.5} \frac{1}{1+x^3}\,dx$ with an error of at most .001 .

AFTER DOING THIS QUESTION PLEASE GET PAGE **A3** TO CHECK YOUR ANSWERS.

4. a) $f(0)$ is a_0, the constant term. So $f(0) = 3$. Also, $a_4 = \frac{f^{(4)}(0)}{4!}$ so $f^{(4)}(0) = 4!\, a_4 = 4!(-19) = -456$.

b) If $f(x) = 3 - 7x + 5x^2 - 19x^4 + \ldots$ then $(f(x))^2 = \left(3 - 7x + 5x^2 - 19x^4 + \ldots\right)^2 =$ (now pause, and think: you are only asked find the first three [non-zero] terms. Why not concentrate on only that task?):

Degree 0 The constant term is $3^2 = 9$.

Degree 1 The only way to get x terms is from the multiplications $3 \cdot (-7x)$ and $(-7x) \cdot 3$, so the degree 1 term in the series for $(f(x))^2$ must be $-42x$.

Degree 2 How can we get x^2 terms? Look at $(3 - 7x + 5x^2)(3 - 7x + 5x^2)$. Let's keep *only* terms of degree exactly 2. The result: $3 \cdot (5x^2) + (-7x) \cdot (-7x) + (5x^2) \cdot 3 = (15 + 49 + 15)x^2 = 79x^2$.

So the first three non-zero terms are $9 - 42x + 79x^2$.

c) If $f(x) = 3 - 7x + 5x^2 - 19x^4 + \ldots$, then $f(x^3) =$ [Substitute x^3 for x everywhere!] $3 - 7(x^3) + 5\left(x^3\right)^2 + \ldots = 3 - 7x^3 + 5x^6 + \ldots$. These are the first three non-zero terms.

d) If $f(x) = 3 - 7x + 5x^2 - 19x^4 + \ldots$ and $g(x) = 8 + 2x - 3x^2 - 7x^3 + \ldots$, then $f(x) + g(x) = 11 - 5x + 2x^2 + \ldots$: I only want the first three terms.

e) Now for the product. $f(x) \cdot g(x) = (3 - 7x + 5x^2 - 19x^4 + \ldots) \cdot (8 + 2x - 3x^2 - 7x^3 + \ldots) = 24 + 3 \cdot (2x) + (-7x) \cdot 8 + 3 \cdot (-3x^2) + (-7x) \cdot (2x) + (5x^2) \cdot 8 + \ldots = 24 - 50x + 17x^2 + \ldots$. The idea is to collect the terms you need, and forget the others.

5. a) Since $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ for $|r| < 1$, $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. The radius of convergence is 1.

b) Since $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ for $|r| < 1$, $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$ for $|x| < 1$. The radius of convergence is 1.

c) Since $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ for $|r| < 1$, $\dfrac{1}{1-x^3} = \sum_{n=0}^{\infty} \left(x^3\right)^n = \sum_{n=0}^{\infty} x^{3n}$ for $|x^3| < 1$ which is the same as $|x| < 1$. The radius of convergence is 1.

d) Since $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ for $|r| < 1$, $\dfrac{1}{1+x^3} = \dfrac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} \left((-x)^3\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{3n}$ for $|x^3| < 1$ which is the same as $|x| < 1$. The radius of convergence is 1.

e) Since $\dfrac{1}{1+x^3} = \sum_{n=0}^{\infty}(-1)^n x^{3n}$ for $|x| < 1$ we can antidifferentiate in that interval. Therefore $\displaystyle\int \frac{1}{1+x^3}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+1} x^{3n+1}$ for $|x| < 1$. What does this series look like? It begins with this: $x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \ldots$.

f) $\displaystyle\int_0^{.5} \frac{1}{1+x^3}\,dx = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \ldots \Big]_0^{.5} = .5 - \frac{1}{4}.5^4 + \frac{1}{7}.5^7 - \frac{1}{10}.5^{10} + \ldots$ but $.5^{10} \approx .00097$ and this is less than the error tolerance. In an alternating series, the sum of the whole series is approximated by a partial sum, with an error which is less than the first omitted term. This means we can approximate the integral by only the first 3 terms: $.5 - \frac{1}{4}(.5)^4 + \frac{1}{7}(.5)^7$.

Comments The decimal value of $.5 - \frac{1}{4}(.5)^4 + \frac{1}{7}(.5)^7$ is approximately $.48549\,10714$ and `Maple` reports that $\displaystyle\int_0^{.5} \frac{1}{1+x^3}\,dx$ is approximately $.48540\,19423$, so the agreement is good. Also, the `Maple` response to the command *int(1/(1+x^3),x);* is the following:

$$\frac{1}{3}\ln(1+\mathrm{x}) = \frac{1}{6}\ln(\mathrm{x}^2 - \mathrm{x} + 1) + \frac{1}{3}\sqrt{3}\arctan\left(\frac{1}{3}(2\mathrm{x}-1)\sqrt{3}\right)$$

which I hope you can recognize as the result of partial fractions painfully applied (although `Maple` didn't complain).

ADVT. $\dfrac{a}{1-r} = \sum_{n=0}^{\infty} ar^n$ if $|r| < 1$. ADVT.

6. a) Find a power series for $\dfrac{1}{1-x}$. What is its radius of convergence?

b) Find a power series for $\dfrac{1}{1+x^2}$. What is its radius of convergence?

c) Integrate the result of part b) to get another power series. What is its radius of convergence?

d) Find a power series for $\arctan x$ centered at 0. What is its radius of convergence?
Hint What is the derivative of $\arctan x$?

e) What is the value of the sixteenth derivative of $\arctan x$ at $x = 0$? What is the value of the seventeenth derivative of $\arctan x$ at $x = 0$?

f) What is $\lim\limits_{x\to 0} \dfrac{\arctan(x^2) - (\arctan x)^2}{x^4}$? Don't use l'Hospital's rule four times! Please don't.

AFTER DOING THIS QUESTION PLEASE GET PAGE **A4** TO CHECK YOUR ANSWERS.

6. a) Since $\dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} ar^n$ for $|r| < 1$, $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$. The radius of convergence is 1.

b) Since $\dfrac{a}{1-r} = \displaystyle\sum_{n=0}^{\infty} ar^n$ for $|r| < 1$, $\dfrac{1}{1+x^2} = \dfrac{1}{1-(-x^2)} = \displaystyle\sum_{n=0}^{\infty} \left((-x)^2\right)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|x^2| < 1$ which is the same as $|x| < 1$. The radius of convergence is 1.

c) Since $\dfrac{1}{1+x^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n}$ for $|x| < 1$ we can antidifferentiate in that interval. Therefore $\displaystyle\int \frac{1}{1+x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ for $|x| < 1$. What does this series look like? It begins like this: $x - \dfrac{1}{3}x^3 + \dfrac{1}{5}x^5 - \dfrac{1}{7}x^7 + \ldots$.

d) Since the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$ and $\arctan 0 = 0$, we know that $\int_0^x \frac{1}{1+x^2}\,dx = \arctan x$. The result in c) tells us that the power series representation for $\arctan x$ centered at 0 is exactly $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \ldots$.

e) The sixteenth derivative would be a_{16} multiplied by 16!. But there are *no* terms of even degree in the power series for $\arctan x$ centered at 0, so $\arctan^{(16)}(0) = 0$. But $a_{17} = \frac{(-1)^8}{2\cdot 8+1}$. This is the coefficient of the term of degree 17, and corresponds to $n = 8$ in the sum. So $\arctan^{(17)}(0)$ must be $(17!) \cdot \frac{1}{17} = 16!$*.

f) Since $\arctan x = x - \frac{1}{3}x^3 + \ldots$, we know that $\arctan(x^2) = x^2 - \frac{1}{3}\left(x^2\right)^3 + \ldots = x^2 - \frac{1}{3}x^6 + \ldots$. Also $(\arctan x)^2 = \left(x - \frac{1}{3}x^3 + \ldots\right)^2 = x^2 - \frac{2}{3}x^4 + \ldots$ so that

$$\arctan\left(x^2\right) - (\arctan x)^2 = x^2 - \frac{1}{3}x^6 + \ldots - \left(x^2 - \frac{2}{3}x^4 + \ldots\right) = \frac{2}{3}x^4 + \ldots.$$

Every use of "…" means "higher-order terms" – terms which have higher degree in x. Therefore the quotient $\dfrac{\arctan\left(x^2\right) - (\arctan x)^2}{x^4}$ must be $\dfrac{2}{3} + \ldots$ and this $\to \dfrac{2}{3}$ as $x \to 0$. So that's the limit.

* `Maple` reports that 16! is 20922789888000 and directly computes the 16^{th} derivative of $\arctan x$ at 0 to be the same number.