Below is a real exam (with some slight changes) given last year at this time in Math 152. Remember that no calculators may be used, and that a formula sheet will be distributed.

1. Find a solution to the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+1}$ that satisfies the initial condition $y(0) = 2$. In the answer express y explicitly as a function of x.

2. The population of a colony of bacteria obeys the exponential law of population growth. Initially the population of the colony is 10^4. After 24 hours, the population is 1.2×10^4.

a) Find the population of the colony as a function of time.

b) What is the population after 30 hours?

3. Evaluate the following sequential limits. Give exact answers, not decimal approximations.

a) $\lim\limits_{n\to\infty} \frac{3\cdot 2^n + 4\cdot n!}{5\cdot 2^n + 3\cdot n!}$ b) $\lim\limits_{n\to\infty} \left(1 - \frac{1}{n}\right)^n$

4. Determine the sum of the following infinite series and express the answer as a rational number, i.e., as the quotient of two integers: $\sum\limits_{n=0}^{\infty} \frac{2\cdot 3^n + (-1)^n}{5^n}$.

5. Test each of the following series for convergence or divergence. State which test has been used and explain why it applies.

a) $\sum\limits_{n=1}^{\infty} \frac{n^{1/2}}{n^2+1}$ b) $\sum\limits_{n=1}^{\infty} \frac{e^n}{n^2 2^n}$ c) $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$

6. Each of the following series converges. How many terms must be included to calculate the sum of each series with error at most $\frac{1}{10^8}$? Explain your answer.

a) $\sum\limits_{n=1}^{\infty} \frac{1}{n^5}$ b) $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^5}$

7. Determine the radius and interval of convergence of each of the following power series. In addition, determine whether each series is absolutely or conditionally convergent at the boundary points of the interval of convergence.

a) $\sum\limits_{n=1}^{\infty} \frac{x^{2n}}{n^2 4^n}$ b) $\sum\limits_{n=0}^{\infty} \frac{2^n x^n}{n!}$

8. Suppose that n is a positive integer.

a) Find the n^{th} Taylor polynomial, $T_n(x)$, of $f(x) = e^x$ centered at $a = 0$.

b) Suppose that $T_n(x)$ is used to approximate e^x at all points of the interval $[0, 1]$. Find n so that the error in using the approximation is at most $\frac{1}{10^6}$ at all points of the interval $[0, 1]$. Explain your answer.

9. Calculate the second Taylor polynomial of $f(x) = (1 + x)^{1/3}$ centered at $a = 0$. Show your work.

10. Determine the Taylor expansion at $a = 0$ of the function $F(x) = \displaystyle\int_0^x \sin(t^3)\, dt$