#13 **Problems for 152:72** 12/4/2001

Below is the first half of a set of problems designed by the principal writer of the final exam in Math 152 this semester to prepare students for this exam. Please learn how to do these problems and similar questions covering the material. A pair of problems is on each **Q** page is followed by their answers on the corresponding **A** page. Certainly you should try to do these problems without notes or text or calculator (exam conditions) but do use these and ask for help if you need it.

1. A region $R$ in the first quadrant is bounded by the line $y = x$ and the parabola $y = -x^2 + 5x - 3$.

a) Find the area of $R$.

b) Find the volume of the solid obtained by rotating $R$ around the $y$-axis.

2. Let $R$ be the first quadrant region bounded by the curves $y = e^x$, $y = e^{-x}$, and the line $x = 1$.

a) What is the area of $R$?

b) What is the volume of the solid obtained by revolving $R$ around the $x$-axis?

c) What is the volume of the solid obtained by revolving $R$ around the $y$-axis?

Please see page **A1** for answers to these two problems.

3. Determine the value of each of the definite integrals. Express the answer in terms of mathematical constants such as $\pi$ or $e$, instead of numerical approximations.

a) $\displaystyle\int_0^{\pi/8} (\cos x)^4\,dx$ b) $\displaystyle\int_0^1 \arctan x\,dx$ c) $\displaystyle\int_0^{\pi/2} (\sin x)^3\,dx$

4. Calculate the following indefinite integrals:

a) $\displaystyle\int \frac{e^{2x}\,dx}{\sqrt{1+e^x}}$ b) $\displaystyle\int (\tan x)^2\,dx$ c) $\displaystyle\int \frac{dx}{(-3+4x-x^2)^{3/2}}$

d) $\displaystyle\int x^3 \ln x\,dx$ e) $\displaystyle\int \frac{x^5}{x^4-1}\,dx$ f) $\displaystyle\int \frac{dx}{\sqrt{x^2+1}}$

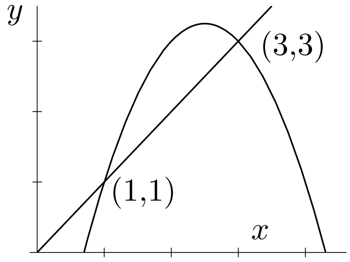
Please see page **A2** for answers to these two problems.

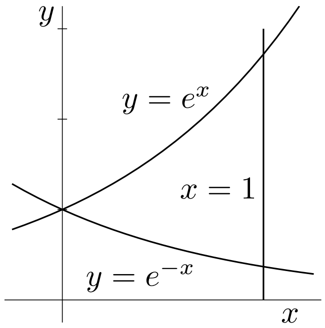
1. a) The parabola $y = -x^2 + 5x - 3$ crosses the line $y = x$ at those points $x$ where $-x^2 + 5x - 3 = x$, or $x^2 - 4x + 3 = 0$. Thus, factoring, $(x-1)(x-3) = 0$, or $x = 1$ and $x = 3$. Thus, the area is: $\int_1^3 ((-x^2 + 5x - 3) - (x))\,dx = \int_1^3 (-x^2 + 4x - 3)\,dx = (-x^3/3 + 2x^2 - 3x)\Big]_1^3 = 4/3$.

b) The volume obtained by rotation of $R$ around the $y$-axis is:

$$2\pi \int_1^3 (x(-x^2 + 5x - 3) - x(x))\,dx = 2\pi \int_1^3 (-x^3 + 4x^2 - 3x)\,dx$$

$$= 2\pi\big(-x^4/4 + 4x^3/3 - 3x^2/2\big)\Big]_1^3 = \frac{16\pi}{3}.$$




2. The upper function is $y = e^x$ and the lower function is $y = e^{-x}$. The curves cross at $x = 0$. A sketch of the region appears at the left.

a) The integral for the area is: $\int_0^1 (e^x - e^{-x})\,dx = e^x + e^{-x}\Big]_0^1 = e + e^{-1} - 2$.

b) The volume obtained by rotation of $R$ around the $x$-axis is:

$$\pi \int_0^1 ((e^x)^2 - (e^{-x})^2)\,dx = \pi \int_0^1 (e^{2x} - e^{-2x})\,dx = \frac{\pi}{2}(e^{2x} + e^{-2x})\Big]_0^1 = \frac{\pi}{2}(e^2 + e^{-2} - 2).$$

2. c) The volume obtained by rotation of $R$ around the $y$-axis is: $2\pi \int_0^1 x(e^x - e^{-x})\,dx$.

Using integration by parts, $\int xe^x\,dx = xe^x - e^x + C$ and $\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$.

Thus, the volume is $2\pi(xe^x - e^x + xe^{-x} + e^{-x})\Big]_0^1 = 2\pi(2e^{-1}) = 4\pi e^{-1}$.

3. a) Since $(\cos x)^2 = \dfrac{1+\cos 2x}{2}$, we know $(\cos x)^4 = \frac{1+2\cos 2x+(\cos 2x)^2}{4}$. Also $(\cos 2x)^2 = \dfrac{(1+\cos 4x)}{2}$ so that $(\cos x)^4 = \dfrac{(3+4\cos 2x+\cos 4x)}{8}$. Therefore $\int\limits_0^{\pi/8} (\cos x)^4\,dx = \frac{1}{8}\int\limits_0^{\pi/8} (3+4\cos 2x+\cos 4x)\,dx = \frac{1}{8}(3x+2\sin 2x+\frac{1}{4}\sin 4x)\Big]_0^{\pi/8} = \dfrac{3\pi+8\sqrt{2}+2}{64}$

b) Use integration by parts: $u = \arctan x$, $dv = dx$, $du = dx/(1+x^2)$, $v = x$. Then $\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln(1+x^2)+C$, so $\int\limits_0^1 \arctan x\,dx = (x\arctan x - \frac{1}{2}\ln(1+x^2))\Big]_0^1 = \frac{\pi-2\ln 2}{4}$.

c) Replace $(\sin x)^2$ by $1-(\cos x)^2$ and substitute: $u = \cos x$, $du = -\sin(x)\,dx$ then $\int\limits_0^{\pi/2} (\sin x)^2\,dx = \int\limits_0^{\pi/2} (1-(\cos x)^2)\sin x\,dx = \int\limits_1^0 -(1-u^2)\,du$, where $u = \cos(\pi/2) = 0$ when $x = \pi/2$, and $u = \cos(0) = 1$ when $x = 0$. This is $-(u-u^3/3)\Big]_1^0 = 1-1/3 = 2/3$.

4. a) Substitute $u = 1+e^x$, $du = e^x dx$, and $e^x = u-1$. Then $\int \frac{e^{2x}\,dx}{\sqrt{1+e^x}} = \int \frac{e^x e^x\,dx}{\sqrt{1+e^x}} = \int \frac{(u-1)du}{\sqrt{u}} = \int (u^{1/2}-u^{-1/2})du = \frac{2}{3}u^{3/2} - 2u^{1/2} + C = \frac{2}{3}(1+e^x)^{3/2} - 2(1+e^x)^{1/2} + C$.

b) Since $(\tan x)^2 = -1+(\sec x)^2$, $\int (\tan x)^2\,dx = \int -1+(\sec x)^2\,dx = -x+\tan x+C$.

c) First complete the square: $x^2-4x+3 = (x-2)^2-1 = u^2-1$ with $u = x-2$. Then substitute $u = x-2$, $du = dx$ so $\int \frac{dx}{(-3+4x-x^2)^{3/2}} = \int \frac{du}{(1-u^2)^{3/2}}$. In this integral, let $u = \sin\theta$, $du = \cos\theta\,d\theta$, $(1-u^2)^{1/2} = \cos\theta$. The integral becomes $\int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int (\sec\theta)^2\,d\theta = \tan\theta + C = \frac{\sin\theta}{\cos\theta} + C = \frac{u}{(1-u^2)^{1/2}} = \frac{x-2}{(-3+4x-x^2)^{1/2}} + C$.

d) Integration by parts with $u = \ln x$, $dv = x^3 dx$, $du = \frac{dx}{x}$, $v = \frac{x^4}{4}$ gives $\int x^3\ln x\,dx = \frac{1}{4}x^4\ln x - \int \frac{x^4}{4}\cdot\frac{dx}{x} = \frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx = \frac{1}{4}x^4\ln x - \frac{1}{16}x^4 + C$.

e) Using long division of polynomials, $x^5 = x(x^4-1)+x$ so $\frac{x^5}{x^4-1} = x + \frac{x}{x^4-1}$. Then we try partial fractions: $\frac{x}{x^4-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}$. Multiplying out and solving for the constants $A, B, C, D$ gives us $\frac{x}{x^4-1} = \frac{1}{4}\left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)$. Antidifferentiation: $\int \frac{x^5\,dx}{x^4-1} = \int x + \frac{1}{4}\left(\frac{1}{x-1} + \frac{1}{x+1} - \frac{2x}{x^2+1}\right)dx = \frac{x^2}{2} + \frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + C$ which is $\frac{x^2}{2} + \frac{1}{4}\ln\left|\frac{x^2-1}{x^2+1}\right| + C$ if you like.

f) Use the substitution $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$, and $(1+x^2)^{1/2} = \sec\theta$ so $\int \frac{dx}{(1+x^2)^{1/2}} = \int \frac{(\sec\theta)^2\,d\theta}{\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|x+\sqrt{1+x^2}\right| + C$.

5. Solve the differential equation $\dfrac{dy}{dx} = (\cos y)^2$ with the initial condition $y(0) = \frac{\pi}{4}$. In the answer, express $y$ explicitly as a function of $x$.

6. A radioactive substance has half-life of 2 years. How long is required for 90% to decay, that is, until only 10% of the original radioactive substance is left?

Please see page **A3** for answers to these two problems.

7. Calculate each of the following improper integrals, or show that the integral is divergent:

a) $\displaystyle\int_0^\infty \frac{\arctan x}{1+x^2}\,dx$ b) $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x+2)}$ c) $\displaystyle\int_0^1 \frac{x\,dx}{(1-x^2)^{3/2}}$

8. Suppose that $a$ is a constant with $a > 1$. Determine the limits of each of the following sequences:

a) $\displaystyle\lim_{n\to\infty} \frac{n^2}{a^n}$ b) $\displaystyle\lim_{n\to\infty} \left(1+\frac{a}{n}\right)^n$ c) $\displaystyle\lim_{n\to\infty} \frac{\ln(a^n+n^2)}{n}$ d) $\displaystyle\lim_{n\to\infty} \left(\sqrt{n^2+an}-n\right)$

Please see page **A4** for answers to these two problems.

5. Separate the variables: $\int dy/(\cos y)^2 = \int dx$. Then $x = \int (\sec y)^2\, dy$ so $x = \tan y + C$. Since $y = \pi/4$ when $x = 0$, $0 = 1 + C$, or $C = -1$. Thus $x = \tan y - 1$, or $\tan y = x + 1$, or $y = \arctan(x + 1)$.

6. Use $R(t) = R_0 e^{kt}$, for constants $R_0$ and $k$, and with $t$ measured in years. Then $R(0) = R_0$ and $R(2) = R_0/2$, since after 2 years half of the original substance is remains. So $R_0 e^{2k} = R_0/2$, or $e^{2k} = 1/2$. Thus, $k = -\ln 2/2$ and $R(t) = R_0 e^{-(\ln 2/2)t}$. If $R(t) = .1R_0$, then $.1R_0 = R_0 e^{-(\ln 2/2)t}$, or $.1 = e^{-(\ln 2/2)t}$. Taking logs (ln's, actually) of both sides, $\ln .1 = -(\ln 2/2)t$, or $-\ln 10 = -(\ln 2/2)t$, or $t = 2\ln 10/\ln 2$.

7. a) Use the substitution $u = \arctan x$, $du = \frac{dx}{1+x^2}$. Change the limits of integration using $\arctan 0 = 0$ and, as $x \to \infty$, $\arctan x \to \frac{\pi}{2}$. The integral $\int_0^\infty \frac{\arctan x}{1+x^2}\,dx$ then becomes $\int_0^{\pi/2} u\,du = \frac{u^2}{2}\Big]_0^{\pi/2} = \frac{\pi^2}{8}$.

b) Use partial fractions: $\frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2}$. Integrating is then easy: $\int_0^\infty \frac{dx}{(x+1)(x+2)} = (\ln|x+1| - \ln|x+2|)\big]_0^\infty = \ln\left|\frac{x+1}{x+2}\right|\Big]_0^\infty = 0 - \ln(1/2) = \ln 2$. We used $\frac{x+1}{x+2} \to 1$ as $x \to \infty$ and therefore $\ln\left|\frac{x+1}{x+2}\right| \to 0$. Thus $\int_0^\infty \frac{dx}{(x+1)(x+2)} = \ln 2$.

c) With the substitution $u = 1 - x^2$, $du = -2x\,dx$ or $x\,dx = -du/2$, we get $\int_0^1 \frac{x\,dx}{(1-x^2)^{3/2}} = \int_1^0 \frac{-du}{2u^{3/2}} = \int_0^1 \frac{du}{2u^{3/2}} = \frac{-1}{u^{1/2}}\Big]_0^1 = \infty$, since as $u \to 0^+$, $1/u^{1/2} \to \infty$. The integral diverges.

8. In the first two replace $n$ by $x$ and use L'Hospital's Rule.

a) The limit is 0. Here, $\lim\limits_{x\to\infty} \frac{x^2}{a^x} = \lim\limits_{x\to\infty} \frac{2x}{\ln a \cdot a^x} = \lim\limits_{x\to\infty} \frac{2}{(\ln a)^2 \cdot a^x} = 0$, since when $a > 1$, $\lim\limits_{x\to\infty} \frac{1}{a^x} = 0$. We used $\frac{d}{dx}a^x = \ln a \cdot a^x$.

b) The limit is $e^a$. To find $\lim\limits_{x\to\infty} (1 + a/x)^x$, let $y = (1 + a/x)^x$. Then find $L = \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \ln\left[(1 + a/x)^x\right] = \lim\limits_{x\to\infty} x \ln(1 + a/x) = \infty \cdot 0$, an indeterminate form. We rewrite the expression so that it has the form $0/0$, to which L'Hospital's Rule applies. Then $L = \lim\limits_{x\to\infty} \frac{\ln(1+a/x)}{1/x} = \lim\limits_{x\to\infty} \left(\left(\frac{-a/x^2}{1+a/x}\right) \Big/ \left(-1/x^2\right)\right) = \lim_{x\to\infty} \frac{a}{1+a/x} = a$. Then $\lim\limits_{x\to\infty} y = \lim\limits_{x\to\infty} e^{\ln(y(x))} = e^L = e^a$.

c) The limit is $\ln a$. Here $\lim\limits_{n\to\infty} \frac{\ln(a^n+n^2)}{n} = \lim\limits_{n\to\infty} \frac{\ln(a^n(1+n^2/a^n))}{n} = \lim\limits_{n\to\infty} \frac{\ln(a^n)+\ln(1+n^2/a^n)}{n} = \lim\limits_{n\to\infty} \frac{n\,\ln(a)}{n} + \lim\limits_{n\to\infty} \frac{\ln(1+n^2/a^n)}{n} = \ln a$, since by part a) of this problem, as $n \to \infty$, $n^2/a^n \to 0$, so $\ln(1 + n^2/a^n) \to \ln 1 = 0$.

d) The limit is $a/2$. We've got to "remember" an algebraic trick. $\lim\limits_{n\to\infty} \left(\sqrt{n^2 + an} - n\right) = \lim\limits_{n\to\infty} \left(\sqrt{n^2 + an} - n\right) \cdot \frac{\left(\sqrt{n^2+an}+n\right)}{\left(\sqrt{n^2+an}+n\right)} = \lim\limits_{n\to\infty} \frac{(n^2+an)-n^2}{\sqrt{n^2+an}+n} = \lim\limits_{n\to\infty} \frac{an}{\sqrt{n^2(1+a/n)}+n} = \lim\limits_{n\to\infty} \frac{an}{n\sqrt{1+a/n}+n} = \lim\limits_{n\to\infty} \frac{a}{\sqrt{1+a/n}+1} = a/2$.