#14 **Problems for 152:72** 12/11/2001

Here is the second half of a set of problems designed by the principal writer of the final exam in Math 152 this semester to prepare students for this exam. Please learn how to do these problems and similar questions covering the material. A pair of problems is on each **Q** page is followed by their answers on the corresponding **A** page. Certainly you should try to do these problems without notes or text or calculator (exam conditions) but do use these and ask for help if you need it.

9. Test the following series for absolute convergence, conditional convergence, or divergence, explaining the test used and how it applies:

a) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n \ln n}{\ln(n^2+1)}$ c) $\displaystyle\sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}$

d) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n\sqrt{\ln n}}$ e) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{2^n}$ f) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{n^2+1}$

10. Determine the radius and interval of convergence of the following power series. In addition, determine those points at which the series is absolutely convergent.

$$\sum_{n=1}^{\infty}\frac{x^{3n}}{n^3 3^n}$$

Please see page **A1** for answers to these two problems.

11. a) Use $\ln(1+x) = \displaystyle\int_0^x \frac{dt}{1+t}$ and term-by-term integration to verify that the Maclaurin series of $\ln(1+x)$ is $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n x^{n+1}}{n+1}$ if $|x|<1$.

b) Use part a) with $x = .5 = \frac{1}{2}$, to approximate $\ln(1.5)$ with $\displaystyle\sum_{n=0}^{10}\frac{(-1)^n}{(n+1)2^{n+1}}$. Give an upper bound (an overestimate) for the error in using this approximation.

12. a) Find the Maclaurin expansion of $F(x) = \displaystyle\int_0^x \arctan(t^2)\,dt$.

b) Use this series to find a numerical series that converges to the integral $\displaystyle\int_0^{1/2}\arctan(t^2)\,dt$.

Please see page **A2** for answers to these two problems.

9. a) Divergent. Use the limit comparison test with the series $\sum_{n=1}^{\infty} \frac{1}{n}$ since $\lim_{n\to\infty} \left(\frac{n}{n^2+1}\right) \Big/ \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$ and since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. The integral test can also be used.

b) Divergent. By L'Hospital, $\lim_{n\to\infty} \frac{\ln n}{\ln(n^2+1)} = \frac{1}{2}$, and so the n^{th} term of the series does not go to 0.

c) Absolutely convergent, using the ratio test: $\frac{|a_{n+1}|}{|a_n|} = \frac{((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^3} = \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} \to \frac{1}{27}$, as $n \to \infty$.

d) Divergent. Use the integral test, with $u = \ln x$, $du = \frac{dx}{x}$: $\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}} = 2\sqrt{\ln x}\,\Big|_2^{\infty} = \infty$.

e) Absolutely convergent, again using the ratio test. Here, $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 \to \frac{1}{2}$, as $n \to \infty$.

f) Conditionally convergent. If $a_n = n/(n^2+1)$, then $a_n > 0$, $\{a_n\}$ is decreasing, and $\lim_{n\to\infty} a_n = 0$. By the alternating series test, the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ converges. By part a) of this problem, the series of absolute values diverges, and thus the series is conditionally convergent.

10. The radius of convergence is $3^{1/3}$ and the interval of convergence is $[-3^{1/3}, 3^{1/3}]$. Here $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{3n+3}}{(n+1)^3 3^{n+1}} \cdot \frac{n^3 3^n}{|x|^{3n}} = \left(\frac{n}{n+1}\right)^3 \cdot \frac{|x|^3}{3} \to \frac{|x|^3}{3}$, as $n \to \infty$. The Ratio Test then implies that the series converges absolutely when $\frac{|x|^3}{3} < 1$, or $|x|^3 < 3$, or $|x| < 3^{1/3}$. The Root Test can also be used to get this answer. When $x = \pm 3^{1/3}$, the series is, respectively, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$, and these series converge absolutely.

11. a) The following geometric series series converges if $|u| < 1$: $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$. Setting $u = -t$, $\dfrac{1}{1+t} = \sum_{n=0}^{\infty} (-1)^n t^n$, if $|t| < 1$. Thus, $\ln(1+x) = \int_0^x \dfrac{dt}{1+t} = \int_0^x \sum_{n=0}^{\infty} (-1)^n t^n \, dt = \sum_{n=0}^{\infty} (-1)^n \int_0^x t^n \, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{n+1}}{n+1}$, if $|x| < 1$.

b) The next term of the series occurs at $n = 11$ and it has absolute value $1/(12 \cdot 2^{12}) = 1/(2^{14}3)$. This is an upper bound for the absolute value of the error.

12. a) Since, if $|x| < 1$, $\arctan x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{2n+1}$, if $|t| < 1$, $\arctan(t^2) = \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+2}}{2n+1}$.
Then $F(x) = \int_0^x \sum_{n=0}^{\infty} \dfrac{(-1)^n t^{4n+2}}{2n+1} \, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2n+1} \int_0^x t^{4n+2} \, dt = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{4n+3}}{(2n+1)(4n+3)}$.

b) Replacing x by $1/2$, the integral is $\sum_{n=0}^{\infty} \dfrac{(-1)^n}{(2n+1)(4n+3)2^{4n+3}}$.

13. a) Find the third Taylor polynomial, $T_3(x)$, of $f(x) = x^{1/4}$ centered at $a = 16$.

b) Use Taylor's inequality to give an upper bound for $|x^{1/4} - T_3(x)|$ when $16 \leq x \leq 18$.

14. The parametric curve $\{x(t) = (\cos t)^3,\ y(t) = (\sin t)^3\}$ is called the *astroid.*

a) Sketch the astroid.

b) Find $\dfrac{dy}{dx}$ as a function of t.

c) Determine the equation of the tangent line to the curve at $t = \pi/6$.

d) Find the arc length of the astroid for $0 \leq t \leq \pi/2$, and the total arc length of the astroid.

15. a) Sketch the polar curve $r = \theta^2$ for $0 \leq \theta \leq \pi$.

b) Find the area bounded by this curve and the x-axis.

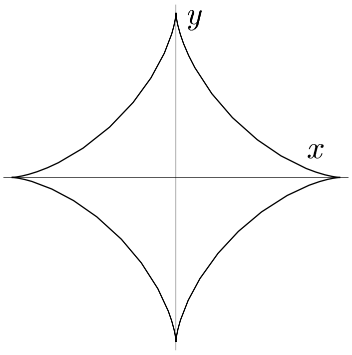
c) Find the arc length of this curve.

Please see page **A3** for answers to these three problems.

13. a) Here $f(x) = x^{1/4}$, $f'(x) = (1/4)x^{-3/4}$, $f''(x) = (-3/4^2)x^{-7/4}$, $f'''(x) = (21/4^3)x^{-11/4}$, and $f^{(4)}(x) = (-231/4^4)x^{-15/4}$. Then $f(16) = 2$, $f'(16) = 1/2^5$, $f''(16) = -3/2^{11}$, and $f'''(16) = 21/2^{17}$ so $T_3(x) = 2 + \frac{1}{2^5}(x-16) - \frac{3}{2^{12}}(x-16)^2 + \frac{7}{2^{18}}(x-16)^3$.

b) Taylor's inequality says that $|f(x) - T_3(x)| \leq M|x-16|^4/4!$ when M is an upper bound for $|f^{(4)}(x)|$ with x in the interval $[16, 18]$. To find an upper bound for $|f^{(4)}(x)|$, we must find x in $[16, 18]$ so that $|f^{(4)}(x)|$ is as large as possible. Since $|f^{(4)}(x)| = (231/4^4)x^{-15/4}$ is decreasing for x in $[16, 18]$, $|f^{(4)}(x)|$ is largest when x is as small as possible, that is, when $x = 16$: $|f^{(4)}(x)| \leq (231/4^4)16^{-15/4} = 231/(4^4{\cdot}16^{15/4})$. Since $16^{1/4} = 2$ and $4^4 = 2^8$, $4^4{\cdot}16^{15/4} = 2^8{\cdot}2^{15} = 2^{23}$. Thus $|f^{(4)}(x)| \leq 231/2^{23}$ and $M = 231/2^{23}$. As x is in $[16, 18]$, $|x-16| \leq 2$, and $|x-16|^4 \leq 2^4$. So we finally get $|f(x) - T_3(x)| \leq \frac{M|x-16|^4}{4!} \leq \left(\frac{231}{2^{23}}\right)\frac{2^4}{2^3 \cdot 3} = \frac{77}{2^{22}}$.

14. a) The graph appears at the right.

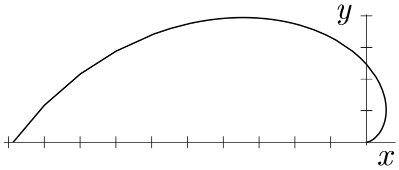


b) Since $\frac{dx}{dt} = -3(\cos t)^2 \sin t$ and $\frac{dy}{dt} = 3(\sin t)^2 \cos t$, $\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{3(\sin t)^2 \cos t}{-3(\cos t)^2 \sin t} = -\tan t$.

c) At $t = \frac{\pi}{6}$, $x = (\cos\frac{\pi}{6})^3 = \frac{3\sqrt{3}}{8}$, $y = (\sin\frac{\pi}{6})^3 = \frac{1}{8}$, and $\frac{dy}{dx} = -\tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}}$. The equation of the tangent line is therefore $y - \frac{1}{8} = \frac{-1}{\sqrt{3}}\left(x - \frac{3\sqrt{3}}{8}\right)$, or $y = \frac{-1}{\sqrt{3}}x + \frac{1}{2}$.

d) $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9(\cos t)^4(\sin t)^2 + 9(\sin t)^4(\cos t)^2 = 9(\sin t)^2(\cos t)^2\left((\cos t)^2 + (\sin t)^2\right) = 9(\sin t)^2(\cos t)^2$. The curve's arc length in the first quadrant is $\int_0^{\pi/2}\sqrt{9(\sin t)^2(\cos t)^2}\,dt = \int_0^{\pi/2} 3\sin t\,\cos t\,dt = \frac{3}{2}(\sin t)^2\Big]_0^{\pi/2} = \frac{3}{2}$. The total arc length is $4 \cdot \left(\frac{3}{2}\right) = 6$.

15. a) The graph appears at the right.



b) The area is: $\int_0^\pi \frac{r^2}{2}\,d\theta = \int_0^\pi \frac{\theta^4}{2}\,d\theta = \frac{\theta^5}{10}\Big]_0^\pi = \frac{\pi^5}{10}$.

c) Since $r = \theta^2$, $\frac{dr}{d\theta} = 2\theta$. The arc length is given by $\int_0^\pi \sqrt{r^2 + (\frac{dr}{d\theta})^2}\,d\theta = \int_0^\pi \sqrt{\theta^4 + 4\theta^2}\,d\theta = \int_0^\pi \theta\sqrt{\theta^2+4}\,d\theta = \frac{(\theta^2+4)^{3/2}}{3}\Big]_0^\pi = \frac{(\pi^2+4)^{3/2} - 8}{3}$.