640:291:01 **Answers to the First Exam** **10/16/2002**

(18) 1. a) Find an equation for the plane tangent to the surface $x^2yz = 6$ at the point $(1,2,3)$.
Answer The gradient is $2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ which at $(1,2,3)$ is $12\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. So we have a normal vector and a point on the plane, and an equation for the plane is $12(x-1) + 3(y-2) + 2(z-3) = 0$.

b) A particle has position vector given by $\mathbf{R}(t) = \cos(2t)\mathbf{i} + \sin(t^2)\mathbf{j} - 3t\mathbf{k}$. Find parametric equations for a line tangent to the path of this particle when $t = 0$.
Answer The velocity vector is $-2\sin(2t)\mathbf{i} + \cos(t^2)2t\mathbf{j} - 3\mathbf{k}$ which at $t = 0$ is $0\mathbf{i} + 0\mathbf{j} - 3\mathbf{k}$. Also note that $\mathbf{R}(0) = 1\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$. Now we have a point on the line and a vector in the direction of the line. So parametric equations for the line are $x = 0t + 1$ and $y = 0t + 0$ and $z = -3t + 0$.

c) The plane found in a) and the line found in b) intersect. Find the point of intersection.
Answer Put the answer to b) in the answer to a) and solve. Get t and then get the point: $12((0t+1) - 1) + 3((0t+0) - 2) + 2((-3t+0) - 3) = 0$. This gives us $t = -2$ so the point we are looking for is $(1,0,6)$. Cheap check: see that $(1,0,6)$ also satisfies the equation for the plane, which it does.

(12) 2. Suppose $\begin{cases} x = u + v^2 \\ y = 2uv - v^3 \end{cases}$. Note that when $u = 3$ and $v = 1$, then $x = 4$ and $y = 5$. a) Suppose u is changed to 3.02 and v is changed to .96. Use linear approximations to estimate the "new" values of x and y.
Answer $x_u = 1$ and $x_v = 2v$ and $y_u = 2$ and $y_v = 2u - 3v^2$, so at $u = 3$ and $v = 1$, $x_u = 1$ and $x_v = 2$ and $y_u = 2$ and $y_v = 3$. Also, $\triangle u = +.02$ and $\triangle v = -.04$. Linearization gives these equations: $\begin{cases} \triangle x \approx x_u\triangle u + x_v\triangle v = 1\cdot(+.02) + 2\cdot(-.04) = -.06 \\ \triangle y \approx y_u\triangle u + y_v\triangle v = 2\cdot(+.02) + 3\cdot(-.04) = -.08 \end{cases}$. Thus the "new" (approximate) values of x and y are 3.94 and 4.92. **Comment** The "true" new values (exactly computed) are 3.9416 and 4.913664.

b) Suppose that we wish to *estimate* what values of u and v near $u = 3$ and $v = 1$ will give the values $x = 4.07$ and $y = 5.03$. Use linear approximations *backwards* to estimate such values.
Answer $\triangle x$ and $\triangle y$ are known (one is .07 and the other is .03). We want to estimate $\triangle u$ and $\triangle v$. They are related by the approximate equations $\begin{cases} .07 \approx 1\triangle u + 2\triangle v \\ .03 \approx 2\triangle u + 3\triangle v \end{cases}$. Treat this as a system of two linear equations in two unknowns. Doubling the first and subtracting it from the second gives us: $-.11 \approx -\triangle v$, so $\triangle v \approx .11$. "Plugging" this value back into the first equation gets us $\triangle u \approx .07 - 2\triangle v \approx .07 - 2\cdot(.11) = -.15$. You can check this, as I did, by substituting the values back into the original linear equations. Thus the new values of u and v are $\begin{cases} \text{new } u = u + \triangle u \approx 3 + (-.15) = 2.85 \\ \text{new } v = v + \triangle v \approx 1 + .11 = 1.11 \end{cases}$. **Comment** If we substitute in the obtained new values for u and v we get 4.0821 for x and 4.959369 for y. I can't explain the (relatively) large discrepancy in y.

(10) 3. Suppose $f(x,y) = x^2y$. Then $f(2,3) = 12$. There is $H > 0$ so that if $\|(x,y) - (2,3)\| < H$ then $|f(x,y) - f(2,3)| < \frac{1}{1{,}000}$. *Find* such an H and explain *why* your assertion is correct.

Answer $|x^2y - 2^2\cdot 3| = |(x^2y - 2^2y) + (2^2y - 2^2\cdot 3)| \overset{\triangle}{\leq} |x^2y - 2^2y| + |2^2y - 2^23|$. I'll do the first part: $|x^2y - 2^2y| = |y||x-2||x+2|$. **IF** $|x-2| < 1$ then $-1 < x - 2 < 1$ so $3 < x + 2 < 5$. We also know that **IF** $|y - 3| < 1$ then $|y| < 4$. We may then (over-)estimate $|x^2y - 1^2y|$ by $4\cdot 5|x-1| = 20|x-1|$. **IF** $|x - 1| < \frac{1}{40{,}000}$ and if the other restrictions (the other two **IF**'s) are fulfilled, this part will be less than $\frac{1}{2{,}000}$ As for the second part: $|2^2y - 2^23| \leq 4|y-3|$. This will be less than $\frac{1}{2{,}000}$ **IF** $|y-3| < \frac{1}{8{,}000}$. So if we take $H = \frac{1}{40{,}000}$, then all four **IF**'s are satisfied, and each of the two parts is less than $\frac{1}{2{,}000}$, so that $|f(x,y) - f(1,2)| < \frac{1}{2{,}000} + \frac{1}{2{,}000} \leq \frac{1}{1{,}000}$. Of course this is not the only valid answer for H.

(12) 4. If $x = s^2 - t^2$, $y = 2st$, and $z = F(x,y)$, show that $\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = 4\sqrt{x^2 + y^2}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$.
Answer I'll use subscripts instead of ∂'s here. The Chain Rule says that $z_s = z_x x_s + z_y y_s = z_x 2s + z_y 2t$. Differentiate this again with respect to s, remembering the Chain Rule (both z_x and z_y may be functions of s and t) and using the product rule: $z_{ss} = (z_x)_s 2s + z_x(2s)_s + (z_y)_s 2t + z_y(2t)_s = (z_{xx}x_s + z_{xy}y_s)2s + 2z_x + (z_{yx}x_s + z_{yy}y_s)2t + z_y(0) = \underline{(4s^2)z_{xx} + (8st)z_{xy}st + (4t^2)z_{yy} + 2z_x}$. Now I do the t derivatives. The Chain Rule again applies: $z_t = z_x x_t + z_y y_t = z_x(-2t) + z_y 2s$. I differentiate once more with respect to t: $z_{tt} = (z_x)_t(-2t) + z_x(-2) + (z_y)_t 2s + z_y(0) = z_{xx}(-2t)^2 + z_{xy}(-2t)(2s) - 2z_x + z_{yx}(-2t)(2s) + z_{yy}(2s)^2 = \underline{(4t^2)z_{xx} - (8st)z_{xy} + (4s^2)z_{yy} - 2z_x}$. The sum of the two underlined expressions, after cancellation and

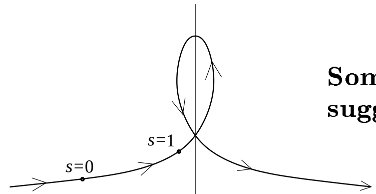
factoring, is $z_{ss}+z_{tt}=4(s^2+t^2)(z_{xx}+z_{yy})$. Since $x=s^2-t^2$ and $y=2st$, $x^2+y^2=s^4-2s^2t^2+t^4+4(st)^2=s^4+2s^2t^2+t^4=(s^2+t^2)^2$, and $\sqrt{x^2+y^2}=s^2+t^2$. So we've verified the requested formula.

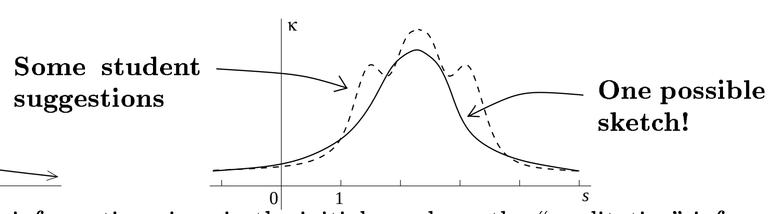
(12) 5. Suppose $F(a,b,c)=a^2b+\sqrt{bc+2c}$. a) Compute $\frac{\partial F}{\partial a}$, $\frac{\partial F}{\partial b}$, and $\frac{\partial F}{\partial c}$ at the point p with coordinates $a=2$, $b=1$, and $c=3$.
Answer $\nabla F(a,b,c)=2ab\mathbf{i}+\left(a^2+\frac{1}{2\sqrt{bc+2c}}c\right)\mathbf{j}+\frac{1}{2\sqrt{bc+2c}}(b+2)\mathbf{k}$. At $(2,1,3)$ we get $4\mathbf{i}+\left(4+\frac{1}{2\cdot3}3\right)\mathbf{j}+\frac{1}{2\cdot3}(1+2)\mathbf{k}$. So $F_a=4$ and $F_b=4\frac{1}{2}$ and $F_c=\frac{1}{2}$.
b) In what direction will F increase most rapidly at p? Write a <u>unit</u> vector in that direction.
Answer $\frac{4\mathbf{i}+4\frac{1}{2}\mathbf{j}+\frac{1}{2}\mathbf{k}}{\sqrt{4^2+\left(4\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$.
c) What is the directional derivative of F at p in the direction found in b)?
Answer $\sqrt{4^2+\left(4\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}$.

(12) 6. Euler investigated the following specific example: $V=x^3+y^2-3xy+\frac{3}{2}x$. He asserted that V has a minimum both at $x=1$ and $y=\frac{3}{2}$ and at $x=\frac{1}{2}$ and $y=\frac{3}{4}$. Was Euler correct?
Answer $V_x=3x^2-3y+\frac{3}{2}$ and $V_y=2y-3x$. Critical points occur where both V_x and V_y are 0. There $2y=3x$ or $y=\frac{3}{2}x$, so the V_x condition becomes: $3x^2-\frac{9}{2}x+\frac{3}{2}=0$ or $6x^2-9x+3=0$ which factors* into $(2x-1)(3x-3)=0$. The critical points *are* as Euler wrote: $(1,\frac{3}{2})$ and $(\frac{1}{2},\frac{3}{4})$. Now test the type of the critical points: $V_{xx}=6x$, $V_{xy}=-3$, $V_{yx}=-3$, $V_{yy}=2$. So the Hessian is $\det\begin{pmatrix}6x & -3\\ -3 & 2\end{pmatrix}=12x-9$. At $(1,\frac{3}{2})$, the Hessian is $12-9>0$, and $V_{xx}=6>0$, so this critical point is a local minimum. At $(\frac{1}{2},\frac{3}{4})$, the Hessian is $12\cdot\frac{1}{2}-9=-3<0$, which makes this critical point a saddle point. Euler was wrong!

(10) 7. A particle has position vector given by $\mathbf{R}(t)=\frac{1}{t}\mathbf{i}+t^2\mathbf{j}-3t\mathbf{k}$.
a) What are the velocity and acceleration vectors of this particle when $t=1$?
Answer $\mathbf{v}(t)=-\frac{1}{t^2}\mathbf{i}+2t\mathbf{j}-3\mathbf{k}$ so $\mathbf{v}(1)=-\mathbf{i}+2\mathbf{j}-3\mathbf{k}$. Also, $\mathbf{a}(t)=\frac{2}{t^3}\mathbf{i}+2\mathbf{j}+0\mathbf{k}$ so $\mathbf{a}(1)=2\mathbf{i}+2\mathbf{j}$.
b) Write the acceleration vector when $t=1$ as a sum of two vectors, one parallel to the velocity vector when $t=1$ and one perpendicular to the velocity vector when $t=1$.
Answer $|\mathbf{v}(1)|=\sqrt{1+4+9}=\sqrt{14}$, and $\mathbf{a}(1)\cdot\mathbf{v}(1)=-2+4=2$ so that $\mathbf{a}_{\|}=\frac{\mathbf{a}(1)\cdot\mathbf{v}(1)}{|\mathbf{v}(1)|^2}\mathbf{v}(1)=\frac{2}{14}(-\mathbf{i}+2\mathbf{j}-3\mathbf{k})$. Normal component: $\mathbf{a}_{\perp}=\mathbf{a}-\mathbf{a}_T=2\mathbf{i}+2\mathbf{j}-\frac{2}{14}(-\mathbf{i}+2\mathbf{j}-3\mathbf{k})$. A check: $\mathbf{a}_{\perp}\cdot\mathbf{v}(1)=\left(\frac{15}{7}\mathbf{i}+\frac{12}{7}\mathbf{j}+\frac{3}{7}\mathbf{k}\right)\cdot(-\mathbf{i}+2\mathbf{j}-3\mathbf{k})=-\frac{15}{7}+\frac{24}{7}-\frac{9}{7}=0$ so that the "normal" component is perpendicular to the velocity vector, as it's supposed to be.

(8) 8. The curve below is parameterized by arc length, s. Arc length is measured forward and backward from the indicated initial point where $s=0$. Sketch a graph of the curvature, $\kappa(s)$, of this curve as well as you can.





Answer There's not much quantitative information given in the initial graph, so the "qualitative" information must be checked. Such a response should show some symmetry around $s\approx 2.2$. I thought it should be "unimodal": increasing before the axis of symmetry, and decreasing after. Some students sketched a graph with "shoulders". I experimented with `Maple` and some models show that such a sketch is possible and even reasonable! At the edges, the curvature should be small. No supporting discussion was requested.

(6) 9. Explain briefly why the following limit does not exist. $\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}$

Answer We look at $(x,y)\neq(0,0)$ here. If $y=x^2$, then $\frac{x^2y}{x^4+y^2}=\frac{x^4}{2x^4}=2$. So the limit along the parabolic path $y=x^2$ as $(x,y)\to(0,0)$ is 2. But along either axis (with $x=0$ or $y=0$), $\frac{x^2y}{x^4+y^2}=0$. Since $0\neq 2$, the limit does not exist.

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* Even then textbook problems were predictable.