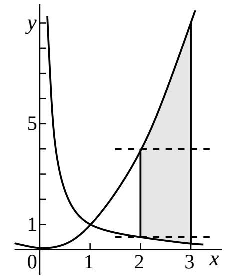
(14) 1. a) Compute $\int_2^3\int_{1/x}^{x^2} x^2y\,dy\,dx$.

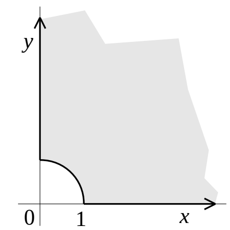
Answer $\int_2^3\int_{1/x}^{x^2} x^2y\,dy\,dx = \int_2^3 \frac{x^2y^2}{2}\Big]_{y=1/x}^{y=x^2}dx = \int_2^3 \frac{x^6}{2} - \frac{1}{2}dx = \frac{x^7}{14} - \frac{x}{2}\Big]_{x=2}^{x=3} = \left(\frac{3^7}{14} - \frac{3}{2}\right) - \left(\frac{2^7}{14} - \frac{2}{2}\right)$. This is a fine answer which `Maple` reports as $\frac{1026}{7}$.

b) Write this iterated integral in $dx\,dy$ order. You may want to begin by sketching the area over which the double integral is evaluated. You are **not** asked to evaluate the $dx\,dy$ result, which may be one or more iterated integrals.



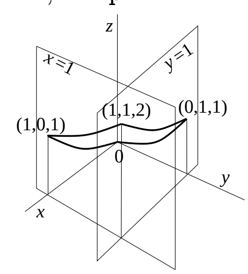
Answer The sketch here (note the different vertical and horizontal scales) shows the (solid) vertical lines $x = 2$ and $x = 3$ and (dashed) horizontal lines $y = \frac{1}{2}$ and $y = 4$ obtained from the intersection of $x = 2$ with the two curves. There will be three $dx\,dy$ integrals, and we will need to "solve" for x as a function of y on the two curves: $x = \frac{1}{y}$ and $x = \sqrt{y}$. The integrals are $\int_4^9\int_{\sqrt{y}}^3 x^2y\,dx\,dy + \int_{1/2}^4\int_2^3 x^2y\,dx\,dy + \int_{1/3}^{1/2}\int_{1/y}^3 x^2y\,dx\,dy$ and I happily report that `Maple` evaluates this to be the same as the answer to a).

(12) 2. Suppose Q is the collection of points in the xy-plane which are both inside the first quadrant and outside the unit circle. Compute $\int\int_Q \frac{1}{(x^2+y^2+1)^3}\,dA$.



Answer Considering the region and the integrand, I believe polar coordinates are appropriate: $\int_0^{\pi/2}\int_1^\infty \frac{1}{(r^2+1)^3}\,r\,dr\,d\theta$. Of course $\int_1^\infty \frac{1}{(r^2+1)^3}\,r\,dr = \lim_{A\to\infty}\int_1^A \frac{1}{(r^2+1)^3}\,r\,dr$ $= \lim_{A\to\infty} \frac{-1}{4(r^2+1)^2}\Big]_1^A = \lim_{A\to\infty} \frac{1}{16} - \frac{1}{4(A^2+1)^2} = \frac{1}{16}$, and finally $\int_0^{\pi/2} \frac{1}{16}\,d\theta = \frac{\pi}{32}$.

(10) 3. Compute the volume of the solid bounded by the xz-plane, the yz-plane, the xy-plane, the planes $x = 1$ and $y = 1$, and the surface $z = x^2 + y^4$.



Answer A sketch may help. The solid has a flat bottom (the xy-plane) and four flat sides (the other planes). It has a curved top which is the graph of $z = x^2 + y^4$, with top "corners" at the origin and at the points $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 2)$. The volume can be computed with a triple iterated integral: $\int_0^1\int_0^1\int_0^{x^2+y^4} 1\,dz\,dy\,dx$ (adding up "$1\,dV$"), or with a double iterated integral: $\int_0^1\int_0^1 x^2 + y^4\,dy\,dx$ (adding up "height dA"). These values are the same. The answer is $\frac{1}{3} + \frac{1}{5} = \frac{8}{15}$.

(16) 4. Find the maximum and minimum values of $F(x, y, z, w) = x + 2y + 3zw$ for points (x, y, z, w) in $\mathbb{R}^4$ satisfying $x^2 + y^2 + z^2 + w^2 = 1$.

Answer Here the constraint $G(x, y, z, w) = x^2 + y^2 + z^2 + w^2 = 1$ means that we restrict our search to the "unit sphere" in $\mathbb{R}^4$. General abstract theory (as mentioned in class) implies that the objective function $F(x, y, z, w) = x+2y+3zw$, a polynomial and therefore continuous, must attain a maximum and a minimum value subject to the constraint. The Lagrange multiplier equations for this problem are the four equations $\lambda\frac{\partial F}{\partial\star} = \frac{\partial G}{\partial\star}$ where $\star$ is each of the variables (x, y, z, and w) together with the constraint equation:

1) $\lambda = 2x$
2) $2\lambda = 2y$
3) $3\lambda w = 2z$
4) $3\lambda z = 2w$
5) $x^2 + y^2 + z^2 + w^2 = 1$

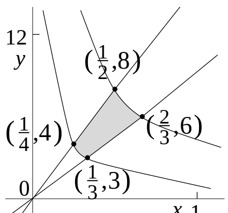
If $\lambda = 0$, all variables are 0, which is not allowed by equation 5). If x or y is 0 then λ is 0 again by equation 1) or equation 2). If either z or w is 0, the other is 0 also using equation 3) or equation 4). Then $x^2 + y^2 = 1$ becomes $\left(\frac{\lambda}{2}\right)^2 + \lambda^2 = 1$, so $\lambda = \pm\frac{2}{\sqrt{5}}$. Therefore $x = \pm\frac{1}{\sqrt{5}}$ and $y = \pm\frac{2}{\sqrt{5}}$ (the signs must be the same) with $z = w = 0$. So consider $\pm(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0, 0)$. F's values are $\pm\frac{5}{\sqrt{5}}$ $= \pm\sqrt{5}$. If all the variables are not 0, divide equation 3) by equation 4) to get $\frac{w}{z} = \frac{z}{w}$ so $z^2 = w^2$. Then $\lambda = \frac{2z}{3w}$ from 3). Since $z = \pm w$, λ must be $\pm\frac{2}{3}$. x must be $\pm_{\mathbf{1}}\frac{1}{3}$ from 1) and y must be $\pm_{\mathbf{1}}\frac{2}{3}$ from 2). Again, x and y have the same sign, indicated by the subscript **1**. Equation 5) now becomes $\frac{1}{9} + \frac{4}{9} + 2z^2 = 1$, so that $z = \pm_{\mathbf{2}}\frac{\sqrt{2}}{3}$ and $w = \pm_{\mathbf{3}}\frac{\sqrt{2}}{3}$. These signs are linked by equations 3) and 4). If $\pm_{\mathbf{1}}$ is $+$, then $\pm_{\mathbf{2}}$ and $\pm_{\mathbf{3}}$ agree. When $\pm_{\mathbf{1}}$ is $-$, they disagree. Therefore we need to examine the values of F on the four points $(\frac{1}{3}, \frac{2}{3}, \pm\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{3})$ and $(-\frac{1}{3}, -\frac{2}{3}, \pm\frac{\sqrt{2}}{3}, \mp\frac{\sqrt{2}}{3})$. These values are $\pm\frac{7}{3}$. Since $\sqrt{5} < \frac{7}{3}$ (because $45 < 49$ [square and multiply]), we have found the maximum and minimum values of F subject to the given constraint.

(16) 5. Integrate the function y over the region in the first quadrant of the xy-plane bounded by the curves $y = \frac{1}{x}$ and $y = \frac{4}{x}$ and $y = 9x$ and $y = 16x$ **using the change of variables technique**. I suggest you try the variables $s = xy$ and $t = \frac{y}{x}$. Describe the corresponding region in the st-plane, compute the area distortion factor (Jacobian), and rewrite the double integral as a $ds\,dt$ integral.

Comment The function, the Jacobian, and the region *should* interact to produce a result easy to deal with, since this is an invented example. The answer is $\frac{14}{3}$.

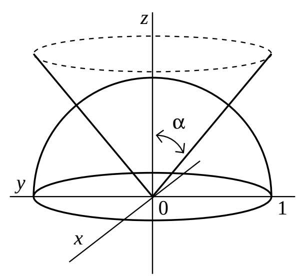
Answer If $s = xy$ and $t = \frac{y}{x}$, then the boundary curves of the region can be described simply. The hyperbolas become $s = 1$ and $s = 4$. The straight lines are $t = 9$ and $t = 16$. Since $st = y^2$, we know that $y = \sqrt{st} = s^{1/2}t^{1/2}$. Then $s = xy = xs^{1/2}t^{1/2}$ so that $x = s^{1/2}t^{-1/2}$. With these formulas, we can compute the Jacobian: $\frac{\partial(x,y)}{\partial(s,t)} = \det\begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} = \det\begin{pmatrix} \frac{1}{2}s^{-1/2}t^{-1/2} & -\frac{1}{2}s^{1/2}t^{-3/2} \\ \frac{1}{2}s^{-1/2}t^{1/2} & \frac{1}{2}s^{1/2}t^{-1/2} \end{pmatrix} = \frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} = \frac{1}{2}t^{-1}$.

This factor multiplies $ds\,dt$ when we "exchange" $dx\,dy$. The st integral is $\int_9^{16}\int_1^4 (s^{1/2}t^{1/2})\,\frac{1}{2}t^{-1}\,ds\,dt$. The $s^{1/2}t^{1/2}$ in parentheses comes from "Integrate the function y over the region …". Multiply and evaluate: $\int_9^{16}\int_1^4 \frac{1}{2}s^{1/2}t^{-1/2}\,ds\,dt = \int_9^{16}\frac{1}{2}t^{-1/2}\,\frac{2}{3}s^{3/2}\Big]_{s=1}^{s=4}\,dt = \frac{7}{3}\int_9^{16} t^{-1/2}\,dt = \frac{7}{3}\,2t^{1/2}\Big]_{t=9}^{t=16} = \frac{14}{3}$.



Irrelevancies The region of integration in the xy-plane is shown, together with intersection points of the boundary curves. Note the different vertical and horizontal scales. *Three* iterated integrals are needed to evaluate the given double integral: $\int_{1/4}^{1/3}\int_{1/x}^{16x} y\,dy\,dx + \int_{1/3}^{1/2}\int_{9x}^{16x} y\,dy\,dx + \int_{1/2}^{2/3}\int_{9x}^{4/x} y\,dy\,dx$. This work took some time. I asked `Maple` for the answer. After correcting *only* two typos (!) $\frac{14}{3}$ was printed. The change of variables technique is preferable.

(12) 6. In this problem H is the upper half of the unit sphere in $\mathbb{R}^3$: those (x, y, z) with $x^2 + y^2 + z^2 \le 1$ and $z \ge 0$. There is a right circular cone whose vertex is $(0, 0, 0)$ and whose axis of symmetry is the positive z-axis which divides the volume of H into two equal parts. Find the angle α that determines this cone. The diagram defines α, which is the angle that the positive z-axis makes with a line on the cone through the vertex.



Answer We set up the volume which is inside both the cone and the unit sphere in spherical coordinates and then compute this volume: $\int_0^1\int_0^{2\pi}\int_0^{\alpha} \rho^2\sin\phi\,d\phi\,d\theta\,d\rho = \frac{2\pi}{3}(1-\cos\alpha)$. When $\alpha = \frac{\pi}{2}$ this is $\frac{2\pi}{3}$. You may know that the volume of a sphere of radius R is $\frac{4\pi}{3}R^3$, so this answer is consistent! Half of the hemisphere's volume is $\frac{\pi}{3}$ which will happen when $\cos\alpha = \frac{1}{2}$, or $\alpha = \frac{\pi}{3}$.

(20) 7. a) Suppose C is the boundary of the unit circle oriented in the usual (counterclockwise) fashion. Compute $\int_C \left(y^2 + \sqrt{1+\cosh(\cos x)}\right)dx + (x + e^{\arctan y})\,dy$.

Answer Here $P(x, y) = y^2 + \sqrt{1+\cosh(\cos x)}$ and $Q(x, y) = x + e^{\arctan y}$ and we will use Green's Theorem applied to the region D which is the inside of the unit circle, whose boundary is C. Then $Q_x - P_y = 1 - 2y$. So the line integral $\int_C P(x, y)\,dx + Q(x, y)\,dy$ equals $\int\int_D (1-2y)\,dA$. The double integral is π because the area of D is π. That's what the 1 in the integrand gives, while the double integral of y over D is 0 since D is symmetric with respect to the x-axis: there's as much positive y as there is negative y. We can also use polar coordinates. The double integral becomes $\int_0^{2\pi}\int_0^1 (1 - r\sin\theta)\,r dr\,d\theta = \int_0^1 (r - r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{3}\sin\theta\,d\theta = \pi$.

b) Suppose D is the path consisting of three straight line segments, first from $(1, 2)$ to $(4, -3)$, then from $(4, -3)$ to $(2, 6)$, and then from $(2, 6)$ to $(3, 4)$. Compute $\int_D (2xy^3)\,dx + (3x^2y^2 + 4y^3)\,dy$.

Answer Here $P(x, y) = 2xy^3$ and $Q(x, y) = 3x^2y^2 + 4y^3$. Direct computation would be intricate: three parameterizations are needed and the functions are not simple. But $P_y = 6xy^2$ and $Q_x = 6xy^2$ are equal and these functions are polynomials. No questions resulting from problems with definition, continuity, or differentiability occur. $P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a gradient vector field: there is $f(x, y)$ with $\nabla f = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. How can we find an f? We can integrate $P(x, y)$ with respect to x (x^2y^3 + a function of y) and compare the result with the integral of $Q(x, y)$ with respect to y ($x^2y^3 + y^4$ + some function of x). These answers represent the same function so we can pick $f(x, y) = x^2y^3 + y^4$ for the potential. The line integral equals $f(3, 4) - f(1, 2)$ (this is $f(\text{THE END}) - f(\text{THE START})$ on the whole "curve", which is the sum of three line segments). The answer is $(3^24^3 + 4^4) - (1^22^3 + 2^4)$ which is a *fine* answer. It equals 808.

(5) Bonus problem: Prove Green's Theorem for the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

Answer Available on the web.