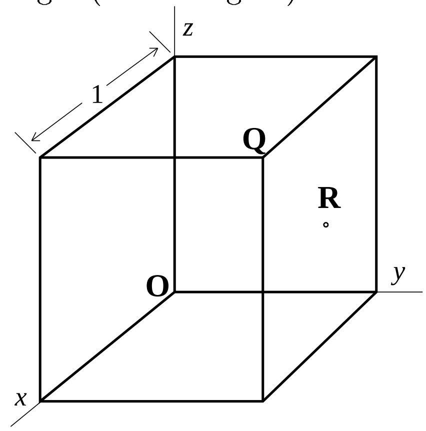
(18) 1. A unit cube lies in the first octant, with a vertex at the origin (see the figure).

a) Express the vectors $\overrightarrow{\mathbf{OQ}}$ (a diagonal of the cube) and $\overrightarrow{\mathbf{OR}}$ (joining **O** to a center of a face not adjacent to **O**) in terms of **i**, **j**, and **k**.

b) Find the cosine of the angle between $\overrightarrow{\mathbf{OQ}}$ and $\overrightarrow{\mathbf{OR}}$.

c) Find an equation for the plane containing **O**, **Q**, and **R**.

This problem was taken in part from an MIT practice exam.

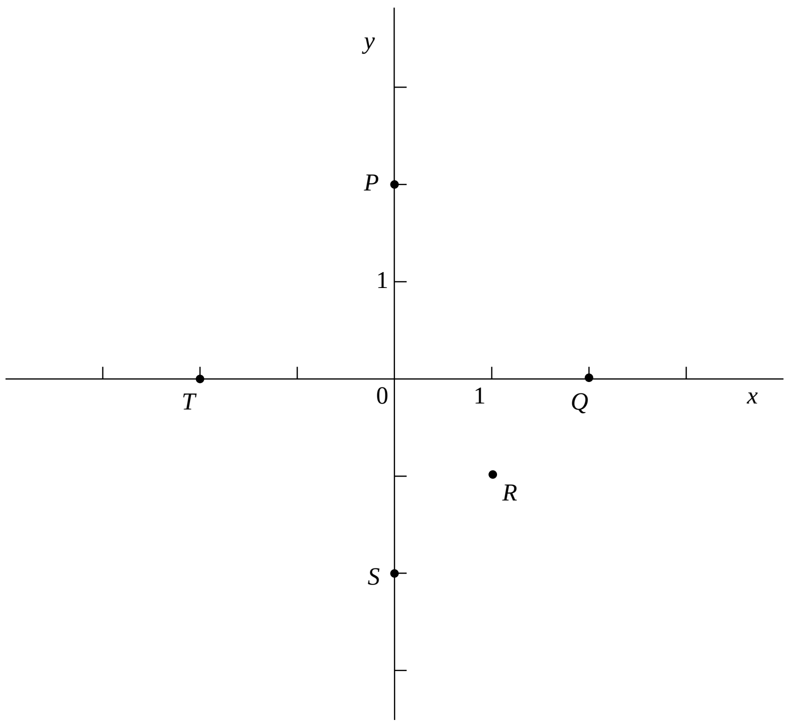


(20) 2. Find all critical points of each function. Describe (as well as you can) the type of each critical point. Explain your conclusions.

a) $f(x,y) = \left(x^2 + y^2\right) e^{x-y}$ b) $g(x,y) = \left(y - x^2\right)^{600}$

(20) 3. Sketch the three level curves of the function $W(x,y) = ye^x$ which pass through the points $P = (0,2)$ and $Q = (2,0)$ and $R = (1,-1)$. **Label each curve** with the **appropriate function value**. Be sure that your drawing is clear and unambiguous.

Also, sketch on the same axes the vectors of the gradient vector field ∇W at the points P and Q and R and S and T. The point $S = (0,-2)$ and the point $T = (-2,0)$.



(12) 4. If $z = f(x,y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\dfrac{\partial z}{\partial r}$ and $\dfrac{\partial z}{\partial \theta}$ and show that

$$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \tfrac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

(10) 5. The point $p = (-2,1,1)$ satisfies the equation $z^3 + xy^2z + 1 = 0$. Suppose near the point p that z is defined implicitly by the equation as a differentiable function of x and y. What is the value of $\dfrac{\partial^2 z}{\partial x^2}$ at p?

(16) 6. Prove Green's Theorem for the region in the plane bounded by the x-axis and the curve $y = 1 - x^2$ by explicitly computing both sides of the equality for a "general" $P(x,y)\,dx + Q(x,y)\,dy$ (be sure to state what conditions on P and Q are needed) and checking that the two sides are indeed the same.

(20) 7. Compute the triple integral of the function $e^{-(x^2+y^2)}$ over the region in the first octant of $\mathbb{R}^3$ which is under the paraboloid $z = 1 - \left(x^2 + y^2\right)$. (The *first octant* in (x,y,z)–space is the collection of points which have $x \geq 0$, $y \geq 0$, and $z \geq 0$.)

Note Be careful: π and e both *definitely* appear in the answer!

(20) 8. Find the total flux upward through the upper hemisphere (where $z \geq 0$) of the sphere $x^2 + y^2 + z^2 = a^2$ of the vector field

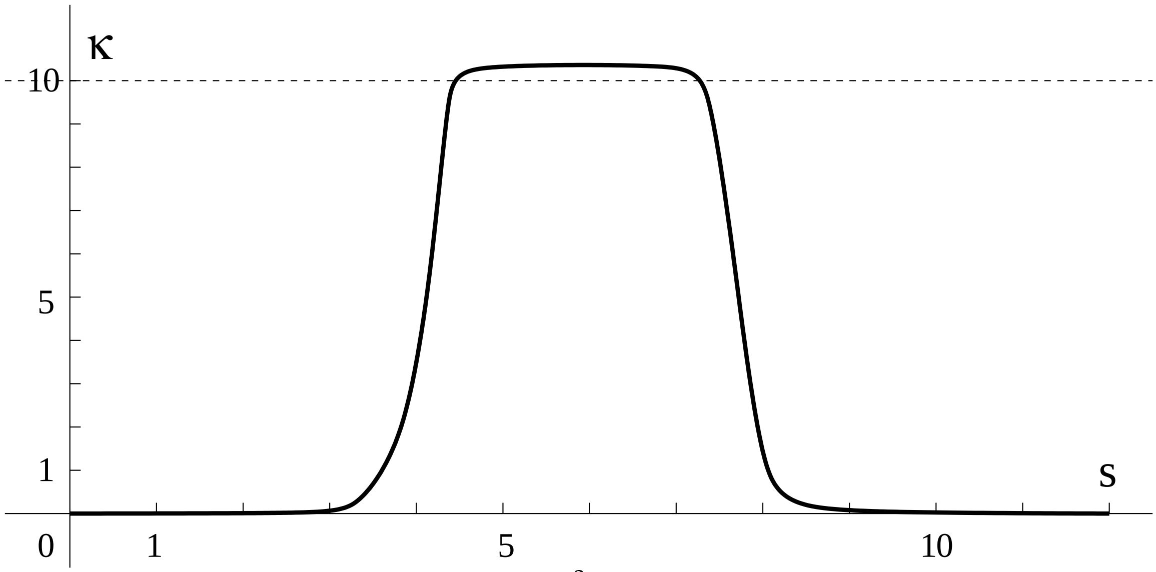
$$\mathbf{T}(x,y,z) = \left(\tfrac{x^3}{3}\right)\mathbf{i} + \left(yz^2 + e^{\sqrt{zx}}\right)\mathbf{j} + \left(zy^2 + y + 2 + \sin\left(x^3\right)\right)\mathbf{k}\,.$$

Note Do *not* attempt to compute this directly! Use the Divergence Theorem on some "simple" solid to change the desired computation to the computation of a triple integral and a much simpler flux integral. Evaluate those integrals, taking as much advantage of symmetry as possible.

(14) 9. Rewrite the integral $\displaystyle\int_0^1\int_0^{2x}\int_0^{3y} H(x,y,z)\,dz\,dy\,dx$ as a $dx\,dy\,dz$ integral.

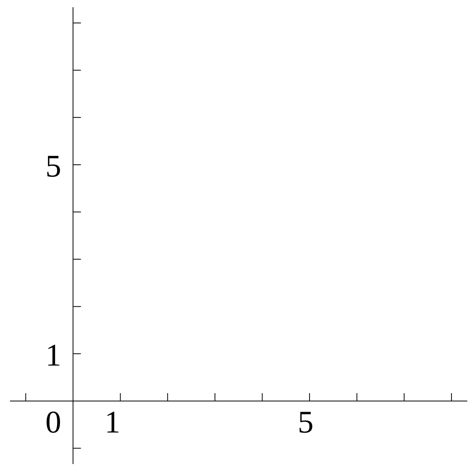
Note Since I have *not* told you what H is, you *cannot* compute the integral. You will probably want to begin by sketching, as accurately as possible, the volume over which the triple iterated integral is taken.

(15) 10. A racetrack for car racing is 12 miles long. The curvature of the road is specified in the graph below. The scale for the horizontal axis (distance along the road, s) is different from the scale for the vertical axis (curvature, κ):



a) Suppose the racetrack is constructed in $\mathbb{R}^2$. Sketch one possible racetrack on the axes to the right.

Answer the following question with a brief explanation: must the racetrack you've sketched cross itself?



b) Now suppose the racetrack can be a curve in space, $\mathbb{R}^3$, with the **same curvature function**. (Imagine a huge roller coaster.) Must such a racetrack intersect itself? Explain your answer as well as you can.

(15) 11. Suppose a, b, and c are positive real numbers so that the point (a, b, c) is on the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$. Prove that the sum of the x-intercept, the y-intercept, and the z-intercept of the tangent plane to $\sqrt{x} + \sqrt{y} + \sqrt{z} = 1$ at (a, b, c) is a constant and does not depend on a, b, or c.

(20) 12. Suppose a vector field is defined by $\mathbf{F}(x, y, z) = \left(y^2 z\right)\mathbf{i} + (2xyz)\,\mathbf{j} + \left(xy^2 + 4z\right)\mathbf{k}$.

a) Determine whether there is a scalar function $P(x, y, z)$ defined everywhere in space such that $\nabla P = \mathbf{F}$. If there is such a P, find it; if there is not, explain why not.

b) Compute the integral $\int_W \mathbf{F} \cdot \mathbf{T}\, ds$, where W is the circular helix whose position vector is given by $\mathbf{R}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k}$ for $0 \le t \le 2\pi$. Use information gotten from your answer to a) to help if you wish.

Final Exam for Math 291, section 1

December 23, 2002

NAME ____

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
You may use a calculator only during the last 20 minutes.
No notes may be used on this exam. A page with formulas will be supplied.

Problem Number	Possible Points	Points Earned:
1	18	
2	20	
3	20	
4	12	
5	10	
6	16	
7	20	
8	20	
9	14	
10	15	
11	15	
12	20	
Total Points Earned:		

Leave answers in "unsimplified" form: $15^2 + (.07) \cdot (93.7)$ **is preferred to** 231.559**. You should know simple exact values of transcendental functions such as** $\cos\left(\frac{\pi}{2}\right)$ **and** $\exp(0)$**. Traditional constants such as** π **and** e **should be left "as is" and not approximated.**

Please check here to get grade information sent to you by e-mail: ____