Formula sheet

to be distributed with the first exam in Math 291, fall 2002

FIRST VERSION 10/14/2002; REVISED 10/15/200

Cauchy-Schwarz: $|v\cdot w| \le \|v\|\,\|w\|$. Triangle $\le$: $\|v+w\| \le \|v\| + \|w\|$.

Distance from $P_0(x_0,y_0,z_0)$ to $P_1(x_1,y_1,z_1)$ is $\sqrt{(x_0-x_1)^2+(y_0-y_1)^2+(z_0-z_1)^2}$.

Distance from $P_1(x_1,y_1,z_1)$ to plane $ax+by+cz=d$ is $\dfrac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$.

Sphere: $(x-h)^2+(y-k)^2+(z-l)^2=r^2$

Plane: $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ where $\mathbf{n}=\langle a,b,c\rangle$

$\|\mathbf{a}\| = \sqrt{(a_1)^2+(a_2)^2+(a_3)^2}$

$\|\mathbf{a}\cdot\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\cos\theta$ (if $=0$, then $\mathbf{a}\perp\mathbf{b}$). $\|\mathbf{a}\times\mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\,\sin\theta$ (if $\mathbf{a}\|\mathbf{b}$ this $=0$.)

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ $\quad\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ $\quad\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$\mathrm{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\|a\|}$ $\quad\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{b}}$

Volume of a parallelepiped with edges $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$: $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$

Arc length: $\int_a^b |\mathbf{r}'(t)|\,dt$ $\quad\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ $\quad\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$ $\quad\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \overset{2\ \mathrm{dim}}{=} \frac{|y''(t)x'\ (t)-x''(t)y'(t)|}{(x'(t)^2+y'(t)^2)^3/2} \overset{y=f(x)}{=} \frac{|f''(x)|}{(1+(f'(x))^2)^{3/2}}$$

$$\tau = \frac{(\mathbf{r}'(t)\times\mathbf{r}''(t))\cdot\mathbf{r}'''(t)}{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|^2}$$ and $d\mathbf{T}/ds=\kappa\mathbf{N}$, $d\mathbf{N}/ds=-\kappa\mathbf{T}+\tau\mathbf{B}$, $d\mathbf{B}/ds=-\tau\mathbf{N}$.

Tangent plane to $z=f(x,y)$ at $P(x_0,y_0,z_0)$: $z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

Linear approx. to $f(x,y)$ at (a,b): $f(x,y)\approx f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$

Tangent plane to $F(x,y,z)=0$:

$$F_x(x_0,y_0,z_0)(x-x_0)+F_y(x_0,y_0,z_0)(y-y_0)+F_z(x_0,y_0,z_0)(z-z_0)=0$$

If y implicitly defined by $y=f(x)$ in $F(x,y)=0$ then $\dfrac{dy}{dx}=-\dfrac{F_x}{F_y}$.

If z implicitly defined by $z=f(x,y)$ in $F(x,y,z)=0$ then $z_x=-\dfrac{F_x}{F_z}$ and $z_y=-\dfrac{F_y}{F_z}$.

$\nabla f = \langle f_x,f_y,f_z\rangle = \dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$ $\quad D_{\mathbf{u}}f(x,y,z)=\nabla f(x,y,z)\cdot\mathbf{u}$

Some chain rules:

If $z=f(x,y)$ and $x=x(t)$ and $y=y(t)$, then $\dfrac{dz}{dt}=\dfrac{\partial f}{\partial x}\dfrac{dx}{dt}+\dfrac{\partial f}{\partial z}\dfrac{dy}{dt}$.

If $z=f(x,y)$ and $x=g(s,t)$ and $y=h(s,t)$, then $\dfrac{\partial z}{\partial s}=\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial s}+\dfrac{\partial f}{\partial y}\dfrac{\partial h}{\partial s}$.

Suppose $f_x(a,b)=0$ and $f_y(a,b)=0$. Let $H=H(a,b)=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$.

a) If $H>0$ and $f_{xx}(a,b)>0$, then $f(a,b)$ is a local minimum.

b) If $H>0$ and $f_{xx}(a,b)<0$, then $f(a,b)$ is a local maximum.

c) If $H<0$, then $f(a,b)$ is not a local maximum or minimum (f has a saddle point).

A real-valued function $F(\mathbf{x})$ is continuous at $\mathbf{x_0}$ if, given any $\varepsilon>0$ there is a $\delta>0$ so that whenever $\|\mathbf{x}-\mathbf{x_0}\|<\delta$, then $|F(\mathbf{x})-F(\mathbf{x_0})|<\varepsilon$.