Answers to review problems for the final exam in Math 291

As of 12/5/2002; answers to 1b) & 3 revised 12/16/2002.

1. Compute the following integrals. Use Green's theorem, Stokes' theorem or the divergence theorem wherever they are helpful.

a) $\iint_D xy\,dA$, where D is the triangle in the xy-plane with vertices (0,0), (2,0), and (0,2).

Answer Convert to an iterated integral: $\int_0^2\int_0^{2-x} xy\,dy\,dx = \int_0^2 x\frac{(2-x)^2}{2}\,dx = \int_0^2 \frac{x^3}{2} - 2x^2 + 2x\,dx = \frac{2}{3}$

b) $\int_C \mathbf{F}\cdot d\mathbf{s}$, where $\mathbf{F}(x,y) = x^2\mathbf{i} - xy\mathbf{j}$ and C is the segment of the parabola $y = x^2$ beginning at $(-1,1)$ and ending at $(1,1)$.

Answer $\begin{cases} x = t \text{ so } dx = dt \\ y = t^2 \text{ so } dy = 2t\,dt \end{cases}$ and $\int_C x^2\,dx - xy\,dy = \int_{-1}^1 t^2\,dt - t^3\,2t\,dt = \int_{-1}^1 t^2 - 2t^4\,dt = -\frac{2}{15}$.

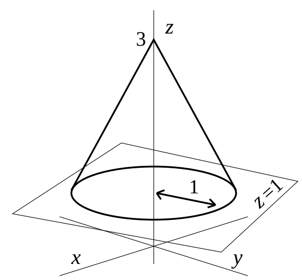
c) $\iint_S (x^3\mathbf{i} + y^3\mathbf{j} + \cos xy\mathbf{k})\cdot\mathbf{n}\,dS$, where S is the unit sphere and $\mathbf{n}$ points inward.

Answer Use the Divergence Theorem. The normal points the "wrong" way, so this integral must equal $-\iiint_R \operatorname{div}(x^3\mathbf{i}+y^3\mathbf{j}+\cos xy\mathbf{k})\,dV = \iiint_R 3x^2+3y^2\,dV$ where R is the region inside the unit sphere. One way to evaluate the triple integral is with cylindrical coordinates (not spherical, because there is no z^2 in the integrand). In iterated spherical coordinates, this becomes $\int_{-1}^1\int_0^{2\pi}\int_0^{\sqrt{1-z^2}}(3r^2)r\,dr\,\theta\,dz$. The initial integral is $\frac{3}{4}r^4\Big]_0^{\sqrt{1-z^2}} = \frac{3}{4}(1-z^2)^2$. Multiply by 2π to compute the next integral, and expand for the last integral. So we have $\int_{-1}^1 \frac{3\pi}{2}(1-2z^2+z^4)dz = \frac{3\pi}{2}(z - \frac{2}{3}z^3 + \frac{1}{5}z^5\Big]_{-1}^1 = \frac{8\pi}{5}$.

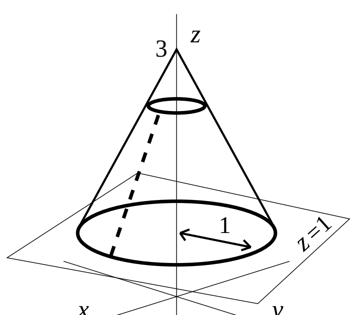
d) $\iint_S z^2\,dS$, where S is the surface $z = \sqrt{x^2+y^2}$, $0 \le z \le 1$.

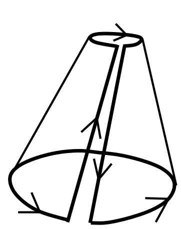
Answer Here $f(x,y) = \sqrt{x^2+y^2}$ and $f_x = \frac{x}{\sqrt{x^2+y^2}}$ and $f_y = \frac{y}{\sqrt{x^2+y^2}}$ so $\sqrt{f_x^2+f_y^2+1} = \sqrt{\frac{x^2+y^2}{x^2+y^2}+1} = \sqrt{2}$ and $\iint_S z^2\,dS = \iint_{\text{The unit circle}}(x^2+y^2)\sqrt{2}\,dA_{xy} \overset{\text{Polar coordinates}}{=} \int_0^{2\pi}\int_0^1 \sqrt{2}(r^2)r\,dr\,d\theta = \frac{\pi}{\sqrt{2}}$.

e) $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz^2\mathbf{j} + z^3\mathbf{k}$ and S is the lateral surface of the cone as shown, with $\mathbf{n}$ pointing outward.



Answer Use Stokes' Theorem. I'll first be a bit cautious. I'd like to apply Stokes' Theorem to the boundary curve which is the circle at the bottom of the cone, where the circle is directed counterclockwise when viewed from high up on the z-axis. But the surface has a corner in the middle. We can't (or, rather, we shouldn't!) ignore questions of existence of derivatives. Simple examples in connection with Green's Theorem [see the class diary entry for 11/20/2002] show that existence of the functions and suitable derivatives at *every* point inside a curve are important. So I'll be a bit careful. I'll make a nice region for Stokes' by "drawing" a dashed line up the side of the cone, connecting a small circle around the top with the circle at the base. I think of what I've done as in the somewhat sloppier picture on the right: there I have "separated" the two sides of the dotted line, and I have drawn directions of integration along the edge curves. Integration along the two line segments which result from the original dotted line will cancel. We did arguments similar to this when using Green's Theorem. The line integral for the circle on top involves integrating three polynomials over a circle whose length $\to 0$ as the circle gets closer to $(0,0,3)$. Since the polynomials are continuous everywhere, the integral of the top circle will $\to 0$ because the length of the circle $\to 0$ and the integrands are bounded. Essentially this discussion is providing the foundation for the declaration that we *can* apply Stokes' Theorem to the whole cone with the bottom boundary circle. Stokes' Theorem tells me that $\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot\mathbf{n}\,dS$, where C is the circle at the base of the cone. But in fact I can apply Stokes' Theorem again to assert that the line integral over





C is the same as the flux integral over the base of the cone (this is the same logic as problem #1 in section 16.8). So I compute the flux integral $\iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ where D is the disc in the plane $z = 1$ inside the unit (xy) circle in *that plane*. We compute $\nabla \times \mathbf{F}$: it is $\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz^2 & z^3 \end{pmatrix} = -2zx\mathbf{i} + y\mathbf{j} + (z^2 - z)\mathbf{k}$. On the plane $z = 1$, this is $-2x\mathbf{i} + y\mathbf{j} + 0\mathbf{k}$. The outward unit normal on the disk (the orientation needed for Stokes' Theorem) is $\mathbf{k}$ so the integrand in the flux computation over the disk, $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$, is just 0. The integral of that function over the disk $x^2 + y^2 \le 1$ is 0.
Another way to complete this problem is to compute the line integral over C. This is (remember that $z = 1$) the integral $\int_C \mathbf{F} \cdot \mathbf{n}\, dS = \int_C (y\mathbf{i} + x\mathbf{j} + \mathbf{k}) \cdot \mathbf{n}\, ds = \int_C -x\, dx + y\, dy$ (see the diary entry for 11/13/2002). Finally, that integral can be computed directly using the usual parameterization for the unit circle ($x = \cos\theta$ and $y = \sin\theta$ etc.) or it can be computed using a standard version of Green's Theorem. The result of either computation is 0 again.

2. Compute $\int_C e^x \sin z\, dx + y^2\, dy + e^x \cos z\, dz$, where C is the oriented curve $\mathbf{x}(t) = (\cos t)^3\, \mathbf{i} + (\sin t)^3\, \mathbf{j} + t\mathbf{k}$, $0 \le t \le \pi/2$. First find a potential function.

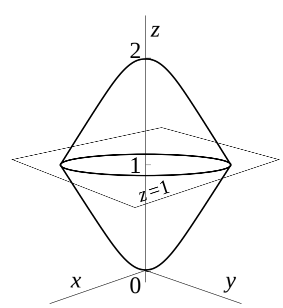
Answer If $\mathbf{V} = e^x \sin z\mathbf{i} + y^2\mathbf{j} + e^x \cos z\mathbf{k}$ then $\nabla \times \mathbf{V} = 0$ is a direct and simple computation. Since the components of the vector field $\mathbf{V}$ are defined in all of $\mathbb{R}^3$, $\mathbf{V}$ must have a potential. We integrate to find it: $\int e^x \sin z\, dx = e^x \sin z + C_1(y, z)$ and $\int y^2\, dy = \frac{1}{3}y^3 + C_2(x, z)$ and $\int e^x \cos z\, dz = e^x \sin z + C_3(x, y)$. The results are three representations of the same function. The three functions $C_\star$ may contain functions only of the variables indicated. We compare the representations and declare that $f(x, y, z) = e^x \sin z + \frac{1}{3}y^3$ is one possible potential. Then $f(\text{THE END}) - f(\text{THE START})$ is the value of the integral over the curve C. When $t = 0$ we get $(1, 0, 0)$ as the position (THE START), and when $t = \pi/2$ we get $(0, 1, \pi/2)$ which is THE END. f's value at THE START is 0 and at THE END it is $\frac{4}{3}$. So the integral's value is $\frac{4}{3}$.

3. A fluid has density 1500 and velocity field $\mathbf{v} = -y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$. Find the flow outward through the sphere $x^2 + y^2 + z^2 = 25$.

Answer Use the Divergence Theorem. The outward flux is given by the "other side" of the Divergence Theorem: $\iiint_V \operatorname{div} \mathbf{v}\, dV$, where V is the region inside the sphere of radius 5 and center $(0, 0, 0)$. Here the divergence of $\mathbf{v}$ is $\frac{\partial(-y)}{\partial x} + \frac{\partial x}{\partial y} + \frac{\partial 2z}{\partial z} = 2$. So we need the integral of 2 over the sphere, and this is twice the volume of the sphere. Thus the flux is $2 \cdot \frac{4}{3}\pi 5^3 = \frac{1000\pi}{3}$. I think (note, please, that this problem was *not* written by me!) that we should multiply the flux by the density to get the total amount of fluid. Of course there are no units mentioned, which would help me, at least! If I am correct, then the answer is $1500 \cdot \frac{1000\pi}{3} = 500{,}000\pi$. Physically I do think this problem makes sense. If the flow were in gallons per minute or something, and if the density were in pounds per gallon, then we would get a product unit of pounds per minute: sensible.

4. Sketch the region E contained between the surfaces $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$ and let S be the boundary of E.

a) Find the volume of E.



Answer Use cylindrical coordinates. I've given a sketch of the surfaces. One is a paraboloid opening up, and the other is a paraboloid opening down. They intersect where $x^2 + y^2 = 2 - x^2 - y^2$, which occurs when $x^2 + y^2 = 1$, so z must be 1. We can compute the volume of E by taking the double integral of the height of the solid over the "base" of the solid. This solid is based over the unit circle in the xy=plane. The height over the point (x, y) is $(2 - x^2 - y^2) - (x^2 + y^2) = 2 - 2x^2 - 2y^2$ (which is THE TOP−THE BOTTOM). I'll convert to polar coordinates. This becomes $\int_0^{2\pi} \int_0^1 (2 - 2r^2) r\, dr\, d\theta$ whose value is π. The volume is π.

b) Let $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Find $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ where $\mathbf{n}$ is the outer normal to S.

Answer Use the Divergence Theorem. The divergence of $\mathbf{F}$ is $\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$, a constant, so that the flux is three times the volume of E, or 3π.