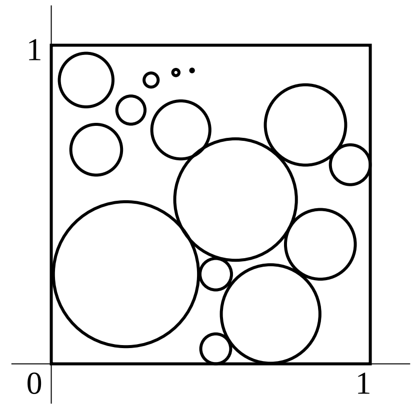
Disconcerting problems about dimensions

Discussion and statement of the first problem

A *sequence of bubbles* is an infinite sequence of circles in the unit square of the plane, $[0,1]\times[0,1]$, whose interiors do not overlap. The center and radius of each circle should be specified in some algebraic or geometric fashion. A picture of some bubbles in one sequence appears to the right.



Is there a sequence of bubbles so that

i the sum of the bubble areas is finite and
ii the sum of the bubble circumferences is infinite?

What you should do

Either give an example of such a sequence of bubbles as explicitly as you can, or explain why no example exists. Your answer should contain a discussion supporting your assertion written in complete English sentences.

Discussion and statement of the second problem

We begin with some terminology and notation.

• $\mathbb{R}^n$ (pronounced "are en") is n-dimensional Euclidean space. A point p in $\mathbb{R}^n$ is an n-tuple of real numbers: $p=(x_1,x_2,\dots,x_n)$. The numbers x_j are called the coordinates of p. For example, $(1,2,-3.8,400,5\pi)$ is a point in $\mathbb{R}^5$.

• If $p=(x_1,x_2,\dots,x_n)$ and $q=(y_1,y_2,\dots,y_n)$ are two points in $\mathbb{R}^n$, the distance from p to q is defined to be $D(p,q)=\sqrt{\sum_{j=1}^n (x_j-y_j)^2}$. This is supposed to be a natural generalization of the usual formulas for distance in $\mathbb{R}^2$ and $\mathbb{R}^3$: a repetition n times of the Pythagorean formula. For example, $p=(1,7,8,-4)$ and $q=(2,-3,9,9)$ are points in $\mathbb{R}^4$, and the distance between them is $\sqrt{(1-2)^2+(7-(-3))^2+(8-9)^2+(-4-9)^2}=\sqrt{271}\approx 16.46208$. The formula for $D(p,q)$ satisfies the usual rules for distances. The text uses $|pq|$ to denote the distance from p to q.

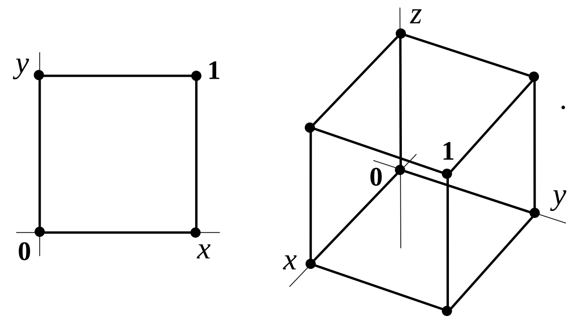
• The origin in $\mathbb{R}^n$ is $\mathbf{0}=(0,0,\dots,0)$, the n-tuple which is all 0's.

• The n-dimensional unit cube is the collection of points $(x_1,x_2,\dots,x_n)$ in $\mathbb{R}^n$ satisfying all of these inequalities: $0\le x_j\le 1$ for $1\le j\le n$.

• The corners of the n-dimensional unit cube are the points $(x_1,x_2,\dots,x_n)$ where each x_j is either 0 or 1. Each of the n choices of the coordinates for a corner can be made independently and there are two alternatives for each coordinate. Therefore the n-dimensional unit cube has 2^n corners.

OVER

Here are some familiar unit cubes, in 2 and 3 dimensions. The corners are marked with $\bullet$'s. The 2-dimensional cube has $2^2 = 4$ corners. The 3-dimensional cube has $2^3 = 8$ corners.



> Do these exercises before starting the problem. The solutions should *not* be handed in! Bare answers ("spoilers") without explanation appear at the bottom of the page. I suggest you look at them *after* you try the problems.

Exercise 1 Suppose $\mathbf{1} = (1, 1, \ldots, 1)$, the n-tuple which is all 1's. Compute the distance between $\mathbf{0}$ and $\mathbf{1}$, which are both corners of the n-dimensional cube. This should convince you that at least *part* of the n-dimensional cube "sticks out" far away from the origin.

Exercise 2 The 20-dimensional unit cube has $2^{20} = 1{,}048{,}576$ corners, far too many to list explicitly. You may need to use a calculator to answer the questions below.

a) How many distinct quadruples of integers are there between 1 and 20? Here order doesn't matter and repetitions aren't allowed: the quadruples $\{4, 7, 13, 17\}$ and $\{7, 4, 17, 13\}$ are "the same" and $\{4, 7, 7, 13\}$ isn't eligible.

b) Use a)'s answer to get a simple <u>overestimate</u> of the total number of corners of the 20-dimensional unit cube which have 1's in <u>at most</u> 4 coordinates. The answer should just be the product of the answer to a) with a fixed number.

c) Use b)'s answer to get an overestimate of the total number of corners of the 20-dimensional unit cube whose distance to $\mathbf{0}$ is at most 2. ($2 = \sqrt{1^2 + 1^2 + 1^2 + 1^2}$.)

d) Use c)'s answer to get an underestimate of the <u>proportion</u> of the corners of the 20-dimensional unit cube which have distance to the origin <u>greater than</u> 2.

You may now believe unit cubes are quite weird when n is large. This is true:

> Suppose A is a positive constant. Define $\#(n, A)$ to be the number of corners of the n-dimensional unit cube whose distance to $\mathbf{0}$ is greater than A. Then
> $$\lim_{n \to \infty} \frac{\#(n, A)}{2^n} = 1$$
> so "almost all" of the corners of the cube are eventually, as dimension grows, farther away from $\mathbf{0}$ than A.

What you should do

Verify the limit statement above. You will use facts from calculus (quote them) about the asymptotic growth of polynomials compared to exponentials. Your answer should contain a discussion supporting your assertion written in complete English sentences.

Hint

Begin with $A = 2$. Follow exercise 2. Generalize the reasoning to $\mathbb{R}^n$ in place of $\mathbb{R}^{20}$. The limit statement for $A = 2$ compares the behavior of a fourth degree polynomial with that of an exponential function. Then verify the statement with $A = 78$. The polynomial's degree now becomes 78^2 but the asymptotics (polynomial growth versus exponential growth) remain qualitatively the same. Please hand in only a report on the general case, if possible.

Answers to exercises: $\sqrt{n}$ | 4845 (Where does this come from?) | 77520 twice (Where does this come from?) | $\approx .92607$