Please write solutions to two of these problems. Hand them in next Wednesday, September 18. These written solutions should be accompanied by explanations using complete English sentences. In this case, all students must work in groups: a group is at least two students and at most four students (recommended size of the group: three). All students should contribute to the writeup and should sign what is handed in.

1. Suppose $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^3$, and the following is known:

$$5 \le \|\mathbf{v}\| \le 10, \quad 30 \le \|\mathbf{w}\| \le 40, \quad 30^\circ \le \theta \le 60^\circ$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$. Then find appropriate over– and under–estimates for the following real numbers using all of the information given:

$$\mathbf{v}\cdot\mathbf{w}, \quad \|\mathbf{v}\times\mathbf{w}\|, \quad \|\mathbf{v}+\mathbf{w}\|\,.$$

2. Suppose $\mathbf{a}$ and $\mathbf{b}$ are vectors, which *you* will choose. Consider the sequence of vectors $\{\mathbf{A}_n\}$ defined recursively by the following conditions:

$$\mathbf{A}_0 = \mathbf{a}; \qquad \mathbf{A}_{n+1} = \mathbf{A}_n \times \mathbf{b} \ \text{ for } n > 0$$

Thus $\mathbf{A}_3 = \big((\mathbf{a}\times\mathbf{b})\times\mathbf{b}\big)\times\mathbf{b}$.

a) Can you choose $\mathbf{a}$ and $\mathbf{b}$ so that all of the sequence $\{\mathbf{A}_n\}$ eventually lies in one plane in $\mathbb{R}^3$? If your answer is "yes", give an example and verify your statement. If your answer is "no", explain why.

b) Can you choose $\mathbf{a}$ and $\mathbf{b}$ so that the sequence $\{\mathbf{A}_n\}$ has a limit in $\mathbb{R}^3$? If your answer is "yes", give an example and verify your statement. If your answer is "no", explain why. (Try some experiments. Interesting examples are better than simple examples.)

c) Can you choose $\mathbf{a}$ and $\mathbf{b}$ so that the sequence $\{\mathbf{A}_n\}$ is eventually periodic? If your answer is "yes", give an example and verify your statement. If your answer is "no", explain why. (Again, interesting examples are better than simple examples.)

3. It is probably true that $(\sin\theta)^2 + (\cos\theta)^2 = 1$. One of those functions has something to do with the $\cdot$ product, and the other has something to do with the $\times$ product. The quoted trig identity implies a *vector formula*. If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbf{R}^3$, <u>state</u> such an identity involving $\mathbf{v}\cdot\mathbf{w}$, $\mathbf{v}\times\mathbf{w}$, and $\|\mathbf{v}\|\,\|\mathbf{w}\|$. Do at least one of the following two tasks:

a) Show how your formula verifies the Cauchy-Schwarz inequality again*.

b) Use `Maple` to <u>prove</u> the formula. (See **More on `Maple`** after the problem statements.)

OVER

* The third proof given of this inequality.

4. **Return to the n-dimensional unit cube**

Advice: Some knowledge of binomial coefficients is useful in this problem.

The notation and abbreviations of the second problem on workshop #1 are continued here. **D** will be the vector along the "main diagonal": its tail is at $(0, 0, \ldots, 0)$ and its head is at $(1, 1, \ldots, 1)$. C will be the right circular cone with angle from the axis $\frac{\pi}{4}$, whose axis is the vector **D**, and whose vertex is at 0. If I_n is number of corners of the cube which lie inside or on the surface of C, prove that

$$\lim_{n \to \infty} \frac{I_n}{2^n} = \frac{1}{2}$$

Hints Suppose **v** is a vector from 0 to a corner. Remember that a corner has coordinates which are either 0 or 1. The corner will be inside the cone exactly when **v** makes an angle $\leq \frac{\pi}{4}$ with **D**. This angle can be computed with the dot product. The computation is not hard, and depends only on the number of 1's in the description of the corner. Then "count" the corners whose **v**'s have the desired angle: the cosine should be $\geq \frac{1}{\sqrt{2}}$. In $\mathbb{R}^3$, the corners inside the cone are $(1, 1, 1)$, $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$.

More on `Maple` Here is information about how to do "vector algebra" using `Maple`. Load the package `linalg`. To do this, enter `with(linalg);` and look at what is printed. Among other things, you will see that `Maple` now knows $\cdot$ which it calls `dotprod` and $\times$ which it calls `crossprod`. Vectors are written `A:=[1,2,3]` and `B:=[4,5,6]`. With these assignments, the statements `dotprod(A,B)` and `crossprod(A,B)` return the values `32` and `[-3,6,-3]` respectively. You can now define `V:=[a,b,c]` and `W:=[p,q,r]` and ask `Maple` to compute and compare (by subtracting) the two sides of the formula. But there's a problem: `Maple` will believe that all the variables involved are complex numbers and for complex numbers the dot product is slightly different. So you need to tell `Maple` that `a`, `b`, ..., `r` are all real numbers. The instructions `assume(a,real)`, `assume(b,real)`, etc. (one for each of the six variables) will do this. I don't know how to do this easily for all variables at once. Have `Maple` do the algebra to verify the vector formula.

> You should hand in relevant pieces of the `Maple` session – both your `Maple` "questions" and the answers. Also write appropriate comments (in complete English sentences!) describing what you have done.