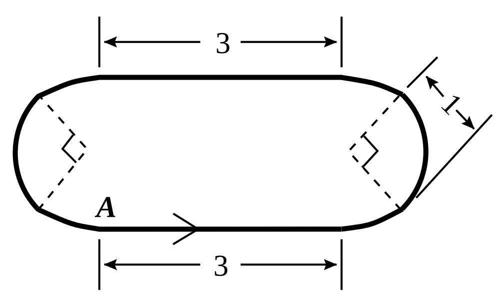
Please write solutions to two of these problems. Hand them in **Monday, February 17**. *The written solutions should be accompanied by explanations using complete English sentences. Students may work alone or* **in pairs**. *Students working in pairs should contribute to the writeup and proofread all of it.*

1. a) Compute the curvature of the plane curve $\begin{cases} x(t) = \dfrac{1-t^2}{1+t^2} \\ y(t) = \dfrac{2t}{1+t^2} \end{cases}$. Explain the result using reasoning independent of the curvature computation.

b) Compute the curvature of the space curve $\begin{cases} x(t) = \dfrac{2-t^2}{1+t^2} \\ y(t) = \dfrac{2t^2-2}{1+t^2} \\ z(t) = \dfrac{3t^2-2}{1+t^2} \end{cases}$. Explain the result using reasoning independent of the curvature computation.

Comment/hint I think I could do the computation in a) "by hand". Probably I could do b) also, but *I would rather get help from a silicon friend.* The cryptic sentences beginning "Explain the result ..." mean that after direct computation of the curvatures, there are ways of verifying the results *independent* of the curvature computations.

2. This picture of an "oval" racetrack has been drawn with some care. Dimensions are in miles. Two portions of the track are straight and two portions are circular arcs, as indicated. Imagine that you drive a car on the track for 20 miles, beginning at the point A in the direction indicated (counterclockwise). Sketch a graph of the curvature, κ, as a function of the distance driven. Briefly and clearly explain your graph.



3. Suppose that $y = f(x)$ is a function whose domain is <u>all of $\mathbb{R}$</u>. Show that the following is *impossible*:

The curvature $\kappa(x)$ at every point of the graph of $y = f(x)$ is at least 1: $\kappa(x) \geq 1$ for all x.

Comment/hint You may first "appreciate" this statement with some sketches (remember, the example $f(x) = x^2$ flattens out towards the edges as $x \to \pm\infty$ so it doesn't violate the impossibility assertion!), but I would like a verification using calculus. One way is to integrate an inequality.

4. a) Prove that the torsion of a space curve which lies in a plane must always be 0.

b) Prove that if the torsion of a space curve is always 0, then the curve must lie in a plane.

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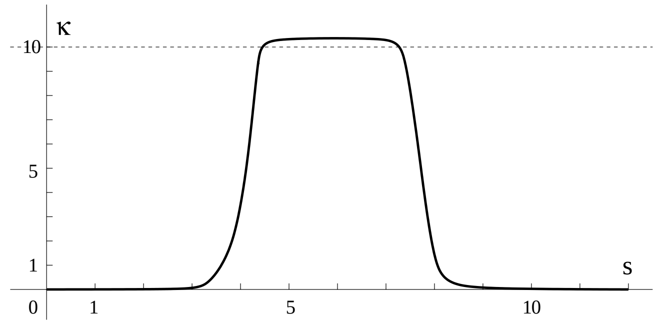
5. Another racetrack* for car racing is 12 miles long. The curvature of the road is specified in the graph displayed to the right. The scale for the horizontal axis (distance along the road, s) is different from the scale for the vertical axis (curvature, κ).



a) Suppose the racetrack is constructed in $\mathbb{R}^2$. Sketch one possible racetrack. Briefly explain the sketch you have drawn, and answer the following question with a brief explanation: must the racetrack you've sketched cross itself?

b) Now suppose the racetrack can be a curve in space, $\mathbb{R}^3$, with the **same curvature function**. (Imagine a huge roller coaster.) Must such a racetrack intersect itself? Explain your answer as well as you can.

6. Find an example of a curve C passing through $p = (0,0,0)$ and $q = (1,1,1)$ so that the curvature and torsion of C at p are, respectively, 3 and 5, and the curvature and torsion of C at q are, respectively, 7 and 11.

7. Suppose C is a curve sitting on the surface of a sphere of radius 1. Verify that the curvature of C can't be smaller than 1.

8. Suppose the curve C has position vector $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t^2\mathbf{k}$.

a) What is the unit tangent vector to C when $t = \pi$?

b) Compute the length of the part of C between $t = 0$ and $t = 2\pi$.

c) Verify that the curvature of this curve is given by the formula $\kappa(t) = \sqrt{\frac{5+4t^2}{(4t^2+1)^3}}$. Even though the first two components of this curve describe uniform circular motion, $\lim_{t\to\infty} \kappa(t) = 0$. Explain briefly why this can happen, using complete English sentences possibly assisted by properly labelled diagrams.

Comment/hint I think I would do the computation in a) "by hand". Probably I could do b) and c) that way also, but *I would rather get help from a silicon friend.*

* Racetracks do *not* necessarily need to be closed curves, or closed courses, or have the end of the track=the beginning of the track.