#6 **Problems for 291:01** 10/21/2002

Please write solutions to two of these problems. Hand them in Monday, October 28. The written solutions should be accompanied by explanations using complete English sentences. Students may work in groups of at most two. They may ask me questions.

1. a) What is the maximum of the function $f(x, y) = 3x + 5y$ subject to the constraint $x^2 + y^2 = 1$, and where is it attained? Draw a picture.

b) What is the maximum of the function $f(x, y) = 3x + 5y$ subject to the constraint $x^{100} + y^{100} = 1$, and where is it attained? Draw a picture.

c) What is the maximum of the function $f(x, y) = 3x + 5y$ subject to the constraint $x^{1/100} + y^{1/100} = 1$, and where is it attained? Draw a picture.

d) What happens to the maximum of the function $f(x, y) = 3x + 5y$ subject to the constraint $x^n + y^n = 1$ and to where it is attained when $n \to +\infty$? Draw a picture.

e) What happens to the maximum of the function $f(x, y) = 3x + 5y$ subject to the constraint $x^n + y^n = 1$ and to where it is attained when $n \to 0^+$? Draw a picture.

2. Doreen and Charlemagne want to build a box with a square base and an open top which will hold 5 ft^3. The bottom and one side are to be built of material which is *twice* as expensive as ... Hold it, hold it! Forget that, and just find the maximum and minimum of $Axy + Byz + Cxz$ subject to the constraint $xyz = 1$ where A, B, and C are all positive.

3. `A real-life problem!` What are the maximum and minimum of $5x+7y$ subject to the restrictions: $y-(.9)x \leq .5$ and $y-(1.2)x \geq .3$ and $y+(2.3)x \geq .7$ and $y-(2.1)x \geq .1$* .

4. Polynomials have roots. For example,

$$(x - 1)(x + 2)(x - 3) = x^3 - 2x^2 - 5x + 6$$

which means that $x^3 - 2x^2 - 5x + 6$ is 0 when $x = 1$ or $x = -2$ or $x = 3$. Low degree polynomials have algebraic recipes for roots in terms of their coefficients. There are no such formulas for higher degree polynomials. If r_1, r_2, and r_3 are the roots of a third degree polynomial, $x^3 + Ax^2 + Bx + C$, then the coefficients (A, B, and C) are functions of the roots (r_1, r_2, and r_3).

a) What are the functions? Verify for the functions you give that if $\begin{cases} r_1 = 1 \\ r_2 = -2 \\ r_3 = 3 \end{cases}$ then $\begin{cases} A = -2 \\ B = -5 \\ C = 6 \end{cases}$.

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* If you can do this without drawing a picture, I admire you. I also regret your taste. This is a precise example of what is called a "linear programming" problem: a rather simple (in some sense) function to be extremized (what are the critical points of this function?) and a domain whose boundary is made up of flat simple pieces. It is, again in some sense, "easy". Real-life problems may have ten or twenty thousand variables and much more than that many inequalities describing the boundary of the domain of the function. It is then not at all easy to describe how to extremize the function subject to the given constraints, even with extensive computer help.

b) Suppose the roots are changed: $\begin{cases} r_1 : 1 \to 1.02 \\ r_2 : -2 \to -2.04 \\ r_3 : 3 \to 2.95 \end{cases}$. Predict the *approximate* changes in the coefficients. (Use partial derivatives and give the linear part of the perturbation; this is easy.)

c) Suppose now that the coefficients are changed. That is, we observe new coefficients: $\begin{cases} A = -2.03 \\ B = -5.02 \\ C = 6.01 \end{cases}$. Approximately what roots would give rise to these coefficients? (This is harder, and you need a *new idea*: what perturbations in the roots will, to first order, give these perturbations in the coefficients? You'll need to solve a "system" of three linear equations in three unknowns: perhaps more arithmetic than you'd like.)

5. It is certainly possible for the set of critical points of a function defined in $\mathbb{R}^3$ to be a point (e.g., $x^2 + y^2 + z^2$) or a line (e.g., $x^2 + y^2$) or a plane (e.g., x^2). Can you create a function $F : \mathbb{R}^3 \to \mathbb{R}$ whose set of critical points is the twisted cubic, $\mathbf{c}(t) = (t, t^2, t^3)$?