640:291:01 **Answers to the First Exam** **3/8/2003**

(12) 1. Suppose that the position of a point in $\mathbb{R}^2$ is given by $\begin{cases} x = e^t \cos t \\ y = e^t \sin t \end{cases}$.

a) Carefully compute the velocity vector, $\mathbf{v}(t)$, and the acceleration vector, $\mathbf{a}(t)$.

Answer $\mathbf{v}(t) = (e^t \cos t - e^t \sin t)\mathbf{i} + (e^t \sin t + e^t \cos t)\mathbf{j}$; $\mathbf{a}(t) = (e^t \cos t - e^t \sin t - e^t \sin t - e^t \cos t)\mathbf{i} + (e^t \sin t + e^t \cos t + e^t \cos t - e^t \sin t)\mathbf{j} = -2e^t \sin t\,\mathbf{i} + 2e^t \cos t\,\mathbf{j}$.

b) Compute the length of the curve from $t = 0$ to $t = 2\pi$.

Answer This is $\int_0^{2\pi} \|\mathbf{v}(t)\|\,dt = \int_0^{2\pi} \sqrt{(e^t \cos t - e^t \sin t)^2 + (e^t \sin t + e^t \cos t)^2}\,dt =$
$\int_0^{2\pi} \sqrt{e^{2t}(\cos t)^2 - 2e^{2t} \cos t \sin t + e^{2t}(\sin t)^2 + e^{2t}(\sin t)^2 + 2e^{2t} \sin t \cos t + e^{2t}(\cos t)^2}\,dt$
$= \int_0^{2\pi} \sqrt{e^{2t} + e^{2t}}\,dt = \int_0^{2\pi} \sqrt{2}e^t\,dt = \sqrt{2}(e^{2\pi} - 1)$.

c) Compute the angle between the position vector and the acceleration vector, and show that the angle does not depend upon t. What is the angle?

Answer $\mathbf{r}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$ so $\mathbf{r}(t)\cdot\mathbf{v}(t) = (e^t \cos t)(-2e^t \sin t) + (e^t \sin t)(2e^t \cos t) = 0$. Therefore $\mathbf{r}(t) \perp \mathbf{a}(t)$: they are always orthogonal. The angle is always $\frac{\pi}{2}$.

(10) 2. a) If $F(x,y) = \frac{3x-4y}{\sqrt{x^2+y^2}}$, briefly explain why $\lim\limits_{(x,y)\to(0,0)} F(x,y)$ does not exist.

Answer As $x \to 0^+$ with $y = 0$ (that is, along the positive x-axis), $F(x,y) = \frac{3x}{x} = 3$. But as $y \to 0^+$ with $x = 0$ (that is, along the positive y-axis), $F(x,y) = \frac{-4y}{y} = -4$. But limits should be *unique*, and since $3 \neq -4$, the limit does not exist.

b) If $G(x,y) = \frac{3x^2-4y^2}{\sqrt{x^2+y^2}}$, briefly explain why $\lim\limits_{(x,y)\to(0,0)} G(x,y)$ exists.

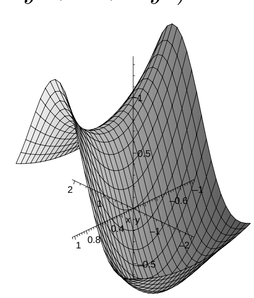
Answer We use polar coordinates here: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ and $G(x,y) = G(r\cos\theta, r\sin\theta) = \frac{3(r\cos\theta)^2 - 4(r\sin\theta)^2}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}}$
$= r\left(3(\cos\theta)^2 - 4(\sin\theta)^2\right)$. As $(x,y) \to (0,0)$, $r \to 0^+$. The parenthesized expression, $\left(3(\cos\theta)^2 - 4(\sin\theta)^2\right)$, is bounded between -4 and 3. The product of a bounded expression and a term approaching 0 must go to 0. So the limit exists and is 0.

(10) 3. Suppose that f is a differentiable function of *one* variable. If $z = f\left(\frac{xy}{x^2+y^2}\right)$ prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

Answer The Chain Rule applies: $\frac{\partial}{\partial x} f\left(\frac{xy}{x^2+y^2}\right) = f'\left(\frac{xy}{x^2+y^2}\right)\frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right)$. But $\frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right) = \frac{y(x^2+y^2)-2x(xy)}{(x^2+y^2)^2}$
$= \frac{-yx^2+y^3}{(x^2+y^2)^2}$. Therefore $x\frac{\partial z}{\partial x} = f'\left(\frac{xy}{x^2+y^2}\right)\left(\frac{-yx^3+xy^3}{(x^2+y^2)^2}\right)$. Now we compute the other part: $\frac{\partial}{\partial y} f\left(\frac{xy}{x^2+y^2}\right) =$
$f'\left(\frac{xy}{x^2+y^2}\right)\frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right)$. So one more computation is needed: $\frac{\partial}{\partial y}\left(\frac{xy}{x^2+y^2}\right) = \frac{x(x^2+y^2)-2y(xy)}{(x^2+y^2)^2} = \frac{-xy^2+x^3}{(x^2+y^2)^2}$. The second part of the partial differential equation's left-hand side is $y\frac{\partial z}{\partial y} = f'\left(\frac{xy}{x^2+y^2}\right)\left(\frac{-xy^3+x^3y}{(x^2+y^2)^2}\right)$. The sum of the parts is 0 since (after adding and factoring what's in common) we have $(-yx^3+xy^3)+(-xy^3+x^3y) = 0$.

(12) 4. Find all critical points of the function $K(x,y) = (y^2+x)e^{(-x^2/2)}$. Describe (as well as you can) the type of each critical point. Explain your conclusions.

Answer Since $K_x = 1e^{-x^2/2} + (y^2+x)e^{-x^2/2}(-x) = (1 - x^2 - xy^2)e^{-x^2/2}$ and $K_y = 2ye^{-x^2/2}$, the critical points are solutions of $\begin{cases} 1 - x^2 - xy^2 = 0 \\ 2y = 0 \end{cases}$ (the exponential is never 0). The c.p.'s are $A = (1,0)$ and $B = (-1,0)$. Now the Hessian: $K_{xx} = (-2x-y^2)e^{-x^2/2} + (1-x^2-xy^2)e^{-x^2/2}(-x) = (-3x-y^2+x^3+x^2y^2)e^{-x^2/2}$, $K_{xy} = (-2xy)e^{-x^2/2}$, $K_{yx} = 2ye^{-x^2/2}(-x) = (-2xy)e^{-x^2/2}$, and $K_{yy} = 2e^{-x^2/2}$. I am happy that my K_{xy} and K_{yx} coincide. All four expressions have the exponential. It is always positive, so it won't affect the sign of the Hessian. I'll delete it in computing H. $H = \det\begin{pmatrix} -3x - y^2 + x^3 + x^2y^2 & -2xy \\ -2xy & 2 \end{pmatrix}$. At A and B, $y = 0$ and $x = \pm 1$, so $H = \det\begin{pmatrix} -3(\pm 1) + (\pm 1) & 0 \\ 0 & 2 \end{pmatrix} = \det\begin{pmatrix} -2(\pm 1) & 0 \\ 0 & 2 \end{pmatrix}$. Therefore, at A, $H = -4$ and we have a saddle point, and, at B, $H = 4$ and $K_{xx} > 0$, so we have a local minimum.



Comment Here's a `Maple` picture of the relevant part of the surface. You may be able to see the saddle and the minimum.

(12) 5. Suppose $f(x,y,z) = x^3 + y^2 z$.

a) Find an equation of the tangent plane for the level surface of f which passes through $(2,1,-3)$.

Answer $\nabla f(x,y,z) = 3x^2\mathbf{i} + 2yz\mathbf{j} + y^2\mathbf{k}$, so $\nabla f(2,1,-3) = 12\mathbf{i} - 6\mathbf{j} + 1\mathbf{k}$ and an equation is $12(x-2) - 6(y-1) + 1(z+3) = 0$.

b) In what direction will f increase most rapidly at $(2,1,-3)$? Write a <u>unit</u> vector in that direction.

Answer The unit vector desired is $\frac{12\mathbf{i}-6\mathbf{j}+1\mathbf{k}}{\sqrt{12^2+6^2+1^2}}$.

c) What is the directional derivative of f at $(2,1,-3)$ in the direction found in b)? **Answer** $\sqrt{12^2+6^2+1^2}$.

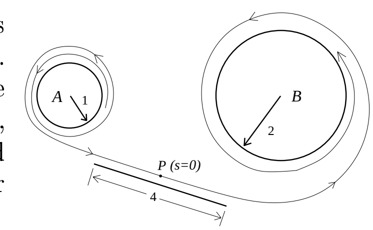
(12) 6. Suppose $f(x,y) = \begin{cases} x & \text{if } x > 0 \\ 2y & \text{if } x \le 0 \text{ and } y > 0 \\ 0 & \text{otherwise} \end{cases}$. a) For which (x,y) in $\mathbb{R}^2$ is f **not** continuous? (Just write your answer carefully. You need *not* give supporting reasons.)

Answer The points to worry about are near the "edges" of the pieces: the x- and y-axes. Considering a few cases gives this answer: f is not continuous for points of the form $(0,t)$ where $t > 0$: the positive y-axis. `Maple` doesn't handle the graph of surfaces with discontinuities well, so I won't show you the result of the commands `f:=(x,y)->piecewise(x>0,x,x<=0 and y>0,2*y,0); plot3d(f(x,y),x=-2..2,y=-2..2);`

b) There is $H > 0$ so that if $\|(x,y) - (0,0)\| < H$ then $|f(x,y) - f(0,0)| < \frac{1}{1{,}000}$. *Find* such an $H > 0$ and explain *why* your assertion is correct. **Note** Any correct $H > 0$ is acceptable, but verification must be given.

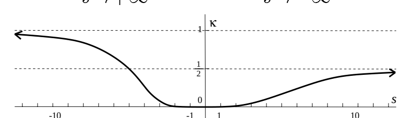
Answer $H = \frac{1}{10{,}000}$ will work. For if $\|(x,y)\| < \frac{1}{10{,}000}$, x and y will both be $< \frac{1}{10{,}000}$, and each of the non-zero pieces of the definition (x and $2y$) will surely be less than $\frac{1}{1{,}000}$. Since $f(0,0) = 0$, this guarantees that the difference $f(x,y) - f(0,0)$ must be less than $\frac{1}{1{,}000}$. Many H's are good answers to this question.

(12) 7. A point is moving along the curve below in the direction indicated. Its motion is parameterized by arc length, s, so that it is moving at unit speed. Arc length is measured from the point P (both backward and forward). The curve is intended to continue indefinitely both forward and backward in s, with its forward motion curling more and more tightly around the indicated circle, B, and, backward, curling more and more tightly around the other circle, A. Near P the curve is parallel to the indicated line segment.



Sketch a graph of the curvature, κ, as a function of the arc length, s. What are $\lim_{s\to+\infty} \kappa(s)$ and $\lim_{s\to-\infty} \kappa(s)$? Use complete English sentences to briefly explain the numbers you give.

Answer I hope to see these features: as $s \to -\infty$, $\kappa \to 1$: the curve is getting closer to a circle of radius 1 with curvature $= 1$. There should be an interval of length ≈ 4 centered at $s = 0$ where the curve is "flat", $\kappa = 0$. As $s \to \infty$, $\kappa \to \frac{1}{2}$: the curve is getting closer to a circle of radius radius 2 with curvature $= \frac{1}{2}$. The graph should decrease and then increase. Also, $\lim_{s\to-\infty} \kappa(s) = 1$ and $\lim_{s\to+\infty} \kappa(s) = \frac{1}{2}$.



Note that the units on the horizontal and vertical axes differ in length.

(12) 8. The polynomial equation $f(x,y,z) = 2xy + x^2 + 5y^3 z + z^4 = 6$ is satisfied by the point $p = (0,-1,2)$ (be careful of the order of the variables – check that this is correct by substituting!). Suppose now that we change the first two coordinates of p and get a point $q = (.03, -1.05, \ldots)$. Use linear approximation to find an approximate value for the third (z) coordinate of q if q also satisfies the equation $f(x,y,z) = 6$.

Answer Since z is the independent variable, we will differentiate $f = 6$ by x and y respectively to get the relevant partial derivatives. $\frac{\partial}{\partial x}$: $2y + 2x + 5y^3\frac{\partial z}{\partial x} + 4z^3\frac{\partial z}{\partial x} = 0$. At $p = (0,-1,2)$ this becomes $-2 + 5(-1)^3\frac{\partial z}{\partial x} + 32\frac{\partial z}{\partial x} = 0$ so that $\frac{\partial z}{\partial x} = \frac{2}{27}$. $\frac{\partial z}{\partial y}$: $2x + 15y^2 z + 5y^3\frac{\partial z}{\partial y} + 4z^3\frac{\partial z}{\partial y} = 0$. At $p = (0,-1,2)$ this becomes $15(-1)^2 2 + 5(-1)^3\frac{\partial z}{\partial y} + 32\frac{\partial z}{\partial y} = 0$ so that $\frac{\partial z}{\partial y} = -\frac{30}{27}$. Therefore $\Delta z \approx \frac{\partial z}{\partial x}\Delta x + \frac{\partial z}{\partial y}\Delta y = \frac{2}{27}(.03) - \frac{30}{27}(-.05)$ and the approximate value of z is 2 + all that.

Comment `Maple` tells me that $2 + \frac{2}{27}(.03) - \frac{30}{27}(-.05) \approx 2.05778$. `fsolve` gives 2.05926, quite close. Please use `implicitplot3d` to examine $f = 6$ when $-5 \le x,y,z \le 5$: you'll see a lovely surface with a hole in it!

(8) 9. The vector $\mathbf{v}$ is $3\mathbf{i} - 7\mathbf{j} + \mathbf{k}$ and the vector $\mathbf{w}$ is $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Write $\mathbf{v}$ as a sum of two vectors, $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$, where $\mathbf{v}_\parallel$ is a scalar multiple of $\mathbf{w}$ and $\mathbf{v}_\perp$ is a vector orthogonal to $\mathbf{w}$.

Answer $\mathbf{v}_\parallel = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\mathbf{w}$, and $\mathbf{v}\cdot\mathbf{w} = 3\cdot 2 - 7\cdot 1 + 1\cdot(-3) = -4$ and $\|\mathbf{w}\|^2 = 2^2 + 1^2 + (-3)^2 = 14$. Therefore $\mathbf{v}_\parallel = -\frac{4}{14}(2\mathbf{i} + \mathbf{j} - 3\mathbf{k}) = -\frac{8}{14}\mathbf{i} - \frac{4}{14}\mathbf{j} + \frac{12}{14}\mathbf{k}$. $\mathbf{v}_\perp = \mathbf{v} - \mathbf{v}_\parallel = 3\mathbf{i} - 7\mathbf{j} + \mathbf{k} - \left(-\frac{8}{14}\mathbf{i} - \frac{4}{14}\mathbf{j} + \frac{12}{14}\mathbf{k}\right) = \frac{50}{14}\mathbf{i} - \frac{94}{14}\mathbf{j} + \frac{2}{14}\mathbf{k}$. Now $\mathbf{w}\cdot\mathbf{v}_\perp = \frac{2\cdot 50 + 1\cdot(-94) + (-3)\cdot 2}{14} = 0$ so maybe this answer is correct.