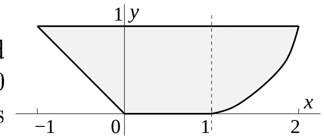
640:291:01 **Answers to the Second Exam** **4/22/2003**

(15) 1. a) Compute $\int_0^1 \int_{-y}^{1+\sqrt{y}} y\,dx\,dy$.

Answer $\int_0^1 \int_{-y}^{1+\sqrt{y}} y\,dx\,dy = \int_0^1 yx\Big]_{x=-y}^{x=1+\sqrt{y}} dy = \int_0^1 y(1+\sqrt{y}) - (y\cdot(-y))\,dy = \int_0^1 y + y^{3/2} + y^2\,dy = \frac{1}{2}y^2 + \frac{2}{5}y^{\frac{5}{2}} + \frac{1}{3}y^3\Big]_{y=0}^{y=1} = \frac{1}{2} + \frac{2}{5} + \frac{1}{3} = \frac{37}{30}$. The last "simplification" is not necessary.

b) Write this iterated integral in $dy\,dx$ order. You may want to begin by sketching the area over which the double integral is evaluated. You are **not** asked to evaluate the $dy\,dx$ result, which may be one or more iterated integrals.



Answer The boundaries of the region include the horizontal lines $y = 0$ and $y = 1$. The inner integral contributes the boundary curves $x = -y$ (for $y \geq 0$ this is a line segment in the second quadrant) and $x = 1 + \sqrt{y}$. For $y \geq 0$ this is a parabola arc in the first quadrant. Now "solve" for y and get $y = (x-1)^2$. Three $dy\,dx$ integrals are needed: $\int_{-1}^0 \int_{-x}^1 y\,dy\,dx + \int_0^1 \int_0^1 y\,dy\,dx + \int_1^2 \int_{(x-1)^2}^1 y\,dy\,dx$. After I corrected three typing errors, `Maple` reported that this value is the same as the answer to a).

(14) 2. Compute the triple integral of $\frac{1}{(x^2+y^2+z^2)^2}$ over the region in $\mathbb{R}^3$ which is in the first octant ($x \geq 0$ and $y \geq 0$ and $z \geq 0$) <u>and</u> outside the unit sphere ($x^2 + y^2 + z^2 = 1$).

Answer In spherical coordinates the region described has $\rho \geq 1$ and both θ and ϕ between 0 and $\frac{\pi}{2}$. So the integral is $\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_1^{\infty} \frac{1}{\rho^4}\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$. The $\rho^2 \sin\phi$ comes from the Jacobian for changing variables into spherical coordinates, and $\frac{1}{\rho^4}$ is the integrand written in spherical coordinates since $\rho^2 = x^2 + y^2 + z^2$. The innermost integral is an improper integral and officially should be evaluated using a limit: $\int_1^{\infty} \frac{1}{\rho^2}\,d\rho = \lim_{A\to\infty} \int_1^A \frac{1}{\rho^2}\,d\rho = \lim_{A\to\infty} -\frac{1}{\rho}\Big]_1^A = \lim_{A\to\infty} 1 - \frac{1}{A} = 1$. Since $\int_0^{\frac{\pi}{2}} \sin\phi\,d\phi = 1$, the triple integral equals $\frac{\pi}{2}$.

(12) 3. Find values of the constants A and B so that the vector field $\mathbf{V} = (x^3 + Ax^2y + y^3)\mathbf{i} + (Bxy^2 - 2x^3)\mathbf{j}$ is a gradient vector field. Find a potential for $\mathbf{V}$ and compute the line integral $\int_C \mathbf{V}\cdot\mathbf{T}\,ds$ when C is a curve starting at (0,2) and ending at $(3,1)$.

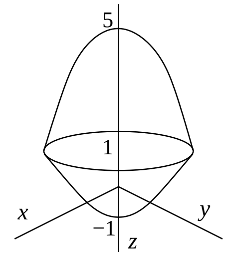
Answer If $P(x,y) = x^3 + Ax^2y + y^3$ and $Q(x,y) = Bxy^2 - 2x^3$ the equation $P_y = Q_x$ becomes $Ax^2 + 3y^2 = By^2 - 6x^2$. Then A must be -6 and B must be 3, so that $\mathbf{V} = (x^3 - 6x^2y + y^3)\mathbf{i} + (3xy^2 - 2x^3)\mathbf{j}$. An antiderivative of P with respect to x is $\frac{1}{4}x^4 - 2x^3y + y^3x + C_1(y)$ and an antiderivative of Q with respect to y is $xy^3 - 2x^3y + C_2(x)$. We compare these two descriptions, and see that a potential for $\mathbf{V}$ is $F(x,y) = \frac{1}{4}x^4 - 2x^3y + xy^3$. Evaluation of the line integral is direct: $\int_C \mathbf{V}\cdot\mathbf{T}\,ds = F(3,1) - F(0,2)$ and $F(0,2) = 0$ while $F(3,1) = \frac{81}{4} - 54 + 3 = -\frac{123}{4}$, which is what's requested.

(16) 4. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x,y,z) = xy^2 + z^4$ for points (x,y,z) in $\mathbb{R}^3$ satisfying $x^2 + y^2 + z^2 = 1$.

Answer Let $g(x,y,z) = x^2 + y^2 + z^2$. The vector equation $\nabla f = \lambda \nabla g$ is the same as the three scalar equations $\begin{cases} y^2 = \lambda(2x) \\ 2xy = \lambda(2y) \\ 4z^3 = \lambda(2z) \end{cases}$. If $\lambda = 0$ then $y = 0$ and $z = 0$ from the first and third equations, and the constraint equation $x^2 + y^2 + z^2 = 1$ then gives as candidates $(\pm 1, 0, 0)$ and there f is 0, which is not likely to be a minimum or maximum value. So we may assume $\lambda \neq 0$. If $x = 0$ then the first equation gives $y = 0$ so the candidates are $(0,0,\pm 1)$ using the constraint again. Now $f(0,0,\pm 1) = 1$, a possible maximum value. $y = 0$ gives $x = 0$ and the same candidates. If $z = 0$ and no other variables are 0, then $\lambda = x$ from the second equation, and the first equation becomes $y^2 = 2x^2$ so that the constraint equation is $3x^2 = 1$ and $x = \pm\frac{1}{\sqrt{3}}$ and $y = \pm\frac{\sqrt{2}}{\sqrt{3}}$. $f(\pm\frac{1}{\sqrt{3}}, \pm\frac{\sqrt{2}}{\sqrt{3}}, 0) = \pm\frac{2}{3\sqrt{3}}$. If no variables are 0, the last equation is $2z^2 = \lambda$, the second is $x = \lambda$ and the first is $y^2 = 2\lambda^2$. So the constraint equation $x^2 + y^2 + z^2 = 1$ becomes $x^2 + 2x^2 + \frac{1}{2}x = 1$ or $3x^2 + \frac{1}{2}x - 1 = 0$ which has as roots $x = \frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}+12}}{6} = \frac{-\frac{1}{2}\pm\sqrt{\frac{49}{4}}}{6} = \frac{-\frac{1}{2}\pm\frac{7}{2}}{6}$. Therefore x is either $\frac{1}{2}$ or $-\frac{2}{3}$. But x can't be negative since $2z^2 = x$. The only acceptable value is $x = \frac{1}{2}$, yielding $y^2 = \frac{1}{2}$ and $z^2 = \frac{1}{4}$. $f(x,y,z) = xy^2 + z^4$ is then $\frac{1}{4} + \frac{1}{16} = \frac{5}{16}$ which is certainly less than 1. The maximum value is 1 and the minimum value is $-\frac{2}{3\sqrt{3}}$.

Comment `Maple` draws wonderful pictures of the sphere and the quite complicated level sets of f.

(12) 5. Find the volume of the finite solid which is between the paraboloids $z = x^2 + y^2 - 1$ and $z = 5 - 2x^2 - 2y^2$.

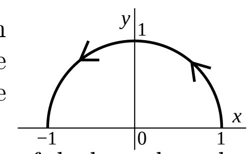


Answer The paraboloids intersect when $x^2 + y^2 - 1 = 5 - 2x^2 - 2y^2$ (the z-values are equal). This occurs when $x^2 + y^2 = 2$, or $z = 1$. For z between -1 (the "bottom") and 1 the sides of the solid are given by $z = x^2 + y^2 - 1$, and for z between 1 and 5 (the "top") the sides are given by $z = 5 - 2x^2 - 2y^2$. The symmetry with respect to the z-axis suggests using cylindrical or polar coordinates to find the volume.

Method I Use double integrals, and treat this as a solid with base the circle of radius $\sqrt{2}$, with the height=Top−Bottom= $(5 - 2x^2 - 2y^2) - (x^2 + y^2 - 1) = 6 - 3(x^2 + y^2) = 6 - 3r^2$. Then the volume is $\int_0^{2\pi}\int_0^{\sqrt{2}}(6 - 3r^2)r\,dr\,d\theta$. The inner integral is $\int_0^{\sqrt{2}} 6r - 3r^3\,dr = 3r^2 - \frac{3}{4}r^4\big]_0^{\sqrt{2}} = 6 - \frac{3}{4}\cdot 4 = 3$. So the value of the double integral is 6π.

Method II Use triple integrals. We will need to split the solid into two pieces: $\int_{-1}^{1}\int_0^{2\pi}\int_0^{\sqrt{z+1}} r\,dr\,d\theta\,dz$ and $\int_1^5\int_0^{2\pi}\int_0^{\sqrt{\frac{5-z}{2}}} r\,dr\,d\theta\,dz$. The value of the first integral is 2π and the value of the second is 4π, so the sum is 6π, the same answer as *Method I.*

(15) 6. Compute $\int_C y\sin(x^3)\,dx + x\,dy$ where C is the upper half of the unit circle, oriented counterclockwise.



Hint Do not attempt to compute this directly. Instead, use Green's Theorem on the upper half of the region inside the unit circle, and also compute a horizontal line integral. Both the area integral and the "other" line integral should be easy. These results then can be used to solve the problem.

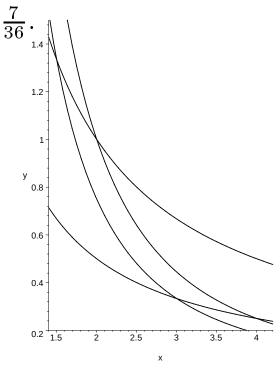
Answer Suppose R is the region inside $x^2+y^2 = 1$ and the x-axis, and let S be the part of the boundary that is on the x-axis (the interval from $x = -1$ to $x = 1$). Then Green's Theorem asserts that $\int\int_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \left(\int_C P\,dx + Q\,dy\right) + \left(\int_S P\,dx + Q\,dy\right)$. The integral over S is 0 since $dy = 0$ and $y = 0$. Since P is $y\sin(x^3)$ and Q is x, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ is $1 - \sin(x^3)$. The integral of 1 over R is $\frac{\pi}{2}$ (the area of half of the unit circle). $\sin(x^3)$ is an odd function of x. That is, it is antisymmetric in x, so its value at $(-x, y)$ is minus its value at (x, y). The integral of $\sin(x^3)$ over R is 0 since R is symmetric with respect to the y-axis. The integral we want to compute is therefore $\frac{\pi}{2}$.

Comment Symbolic computation of the integral with `Maple`, using $\begin{cases} x = \cos t \\ y = \sin t \end{cases}$, is *not* successful!

(16) 7. Integrate the function y over the region in the first quadrant of the xy-plane bounded by the curves $xy = 1$, $xy = 2$, $x^2y = 3$ and $x^2y = 4$ **using the change of variables technique**. I suggest you try the variables $s = xy$ and $t = x^2y$. Describe the corresponding region in the st-plane, compute the area distortion factor (Jacobian), and rewrite the double integral as a $ds\,dt$ integral.

Comment The function, the Jacobian, and the region *should* interact to produce a result easy to deal with, since this is an invented example. The answer is $\frac{7}{36}$.

Answer The region is described by $1 \le s \le 2$ and $3 \le t \le 4$. Now I'll describe x and y in terms of s and t. Divide the equation defining t by the equation defining s to get $x = \frac{t}{s}$. Then $y = \frac{s^2}{t}$ because $xy = s$. The Jacobian is the absolute value of this determinant: $\frac{\partial(x,y)}{\partial(s,t)} = \det\begin{pmatrix} x_s & x_t \\ y_s & y_t \end{pmatrix} = \det\begin{pmatrix} -\frac{t}{s^2} & \frac{1}{s} \\ \frac{2s}{t} & -\frac{s^2}{t^2} \end{pmatrix} = -\frac{1}{t}$. So the Jacobian is $\frac{1}{t}$. The integral becomes $\int_3^4\int_1^2 \left(\frac{s^2}{t}\right)\left(\frac{1}{t}\right) ds\,dt = \int_3^4\int_1^2 \frac{s^2}{t^2}\,ds\,dt$ which is $\frac{7}{36}$.



Slightly relevant Here's a `Maple` picture of the region. Three integrals are needed for an xy computation and the intersection points must be determined. The result is $\int_{3/2}^{2}\int_{3/x^2}^{2/x} y\,dy\,dx + \int_2^3\int_{3/x^2}^{4/x^2} y\,dy\,dx + \int_3^4\int_{1/x}^{4/x^2} y\,dy\,dx$. `Maple` reports this is also $\frac{7}{36}$. Probably the st method is easier.