Finding a nice parameterization of the curve is important. Generally when two surfaces, even two nicely described surfaces, intersect, finding a good parameterization of the intersection curve can be difficult. Here the equation $x^2 + y^2 = 1$ provides a valuable start. I'll try $x = \cos t$ and $y = \sin t$. Then $z = y^2$ tells us that $z = (\sin t)^2$. So the parameterization as a "position vector is $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin t)^2\mathbf{k}$.

Now we need to plan for the computation. It will be **easy** to compute $\mathbf{r}'$ and $\mathbf{r}''$ and $\mathbf{r}'''$ and find their values when $t = \frac{\pi}{4}$ which corresponds to the point $p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$. After I get these three derivatives, I will substitute $t = \frac{\pi}{4}$ to get specific vectors, and use those in the formulas below:

$$\mathbf{T}(\tfrac{\pi}{4}) = \frac{\mathbf{r}'(\pi/4)}{\|\mathbf{r}'(\pi/4)\|} \qquad \kappa(\tfrac{\pi}{4}) = \frac{\|\mathbf{r}'(\pi/4)\times\mathbf{r}''(\pi/4)\|}{\|\mathbf{r}'(\pi/4)\|^3} \qquad \tau(\tfrac{\pi}{4}) = \frac{(\mathbf{r}'(\pi/4)\times\mathbf{r}''(\pi/4))\cdot\mathbf{r}'''(\pi/4)}{\|\mathbf{r}'(\pi/4)\times\mathbf{r}''(\pi/4)\|^2}$$

This is all straightforward. But how about $N(\frac{\pi}{4})$? I don't think there is any way to avoid **some pain** here. We need to compute $\mathbf{T}(t)$ and differentiate it with respect to t (a use of the quotient rule), and then divide by the magnitude of the result. But we only need to do this at $\frac{\pi}{4}$. When we get $\mathbf{N}(\frac{\pi}{4})$, then $\mathbf{B}(\frac{\pi}{4}) = \mathbf{T}(\frac{\pi}{4}) \times \mathbf{N}(\frac{\pi}{4})$, another cross-product.

The Computation

$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (\sin t)^2\mathbf{k}$, so $\mathbf{r}'(t) = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \overbrace{2(\sin t)(\cos t)}^{\sin(2t)}\mathbf{k}$. Therefore $\mathbf{r}'(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + 1\mathbf{k}$ and $\|\mathbf{r}'(t)\| = \sqrt{1 + (\sin(2t))^2}$ (this isn't so bad) and $\|\mathbf{r}'(\frac{\pi}{4})\| = \sqrt{2}$ Also $\mathbf{r}''(t) = (-\cos t)\mathbf{i} + (-\sin t)\mathbf{j} + 2\cos(2t)\mathbf{k}$ so that $\mathbf{r}''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + 0\mathbf{k}$. Finally, $\mathbf{r}'''(t) = (\sin t)\mathbf{i} + (-\cos t)\mathbf{j} - 4\sin(2t)\mathbf{k}$. with $\mathbf{r}'''(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} - 4\mathbf{k}$.

Now for **the easy stuff**:

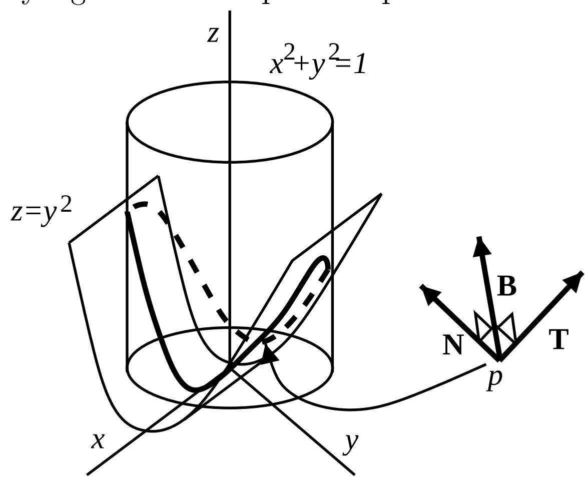
$\mathbf{T}(\frac{\pi}{4}) = \frac{-\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} + 1\mathbf{k}}{\sqrt{2}} = -\frac{1}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$ (a unit vector!). For κ we need $\mathbf{r}'(\pi/4)\times\mathbf{r}''(\pi/4) =$ $\det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \mathbf{k}$ which has magnitude $\sqrt{2}$. So divide this by $\|\mathbf{r}'(\frac{\pi}{4})\|^3 = 2^{3/2}$ and get $\kappa(\frac{\pi}{4}) = \frac{1}{2}$. The "top" of τ is $\left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + \mathbf{k}\right) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} - 4\mathbf{k}\right)$ $= \frac{1}{2} + \frac{1}{2} - 4 = -3$ and the bottom is 2, so the answer seems to be $-\frac{3}{2}$.

And now for **some pain**:

The "pain" is differentiation of $\mathbf{T}(t) = \frac{(-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \sin(2t)\mathbf{k}}{\sqrt{1+(\sin(2t))^2}}$. I will write this as $\frac{\text{Top}}{\text{Bottom}}$, so that its derivative is $\frac{(\text{Top}')(\text{Bottom}) - (\text{Bottom}')(\text{Top})}{(\text{Bottom})^2}$, which I must compute at $t = \frac{\pi}{4}$. Let's see: $(\text{Bottom})^2 = 2$, and $\text{Bottom}' = \frac{1}{2}(1 + (\sin(2t))^2)^{-1/2} 2\sin(2t)\cos(2t)2$ which is 0 when $t = \frac{\pi}{4}$ (I *think* this is correct – nice work with the Chain Rule!). $(\text{Top})'$ is $\mathbf{r}''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + 0\mathbf{k}$, so $\mathbf{T}'(\frac{\pi}{4}) = -\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 0\mathbf{k}$. Adjusting the length to get a unit vector, $\mathbf{N}(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + 0\mathbf{k}$. Finally, $\mathbf{B}(\frac{\pi}{4}) = \mathbf{T}(\frac{\pi}{4}) \times \mathbf{N}(\frac{\pi}{4}) = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} =$ $\frac{1}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$, which is also a unit vector, as "theory" predicts.

OVER

The frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is "orthonormal": three unit vectors, mutually perpendicular. I checked this! The curvature of a space curve is always nonnegative as I explained in class. The torsion may have any sign. Here is a possible picture of the situation:



The curve of intersection of the circular cylinder $x^2+y^2 = 1$ and the parabolic cylinder $z = y^2$ is shown approximately and the point $p = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$ is indicated, along with a possible picture of the Frenet frame at p.