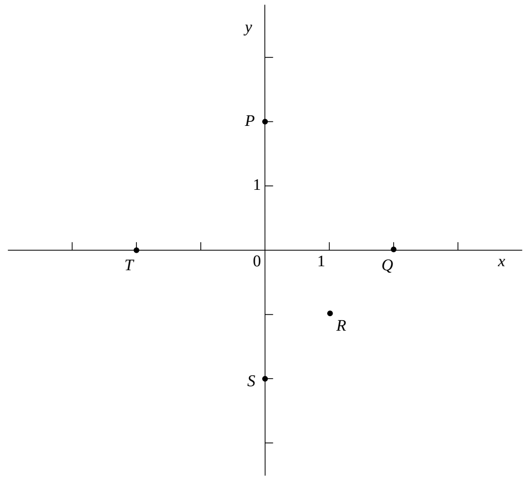
(20) 1. Sketch the three level curves of the function $W(x,y) = \frac{y}{x^2+1}$ which pass through the points $P = (0,2)$ and $Q = (2,0)$ and $R = (1,-1)$. **Label each curve** with the **appropriate function value**. Be sure that your drawing is clear and unambiguous.
Also, sketch on the same axes the vectors of the gradient vector field ∇W at the points P and Q and R and S and T. The point $S = (0,-2)$ and the point $T = (-2,0)$.



(18) 2. Suppose $f(x,y,z) = \sqrt{x + 3y^2 - 2z}$. Then $f(5,1,2) = 2$.

a) Find a linear approximation to $f(5.01, .98, 2.03)$.
You are *not* asked for an exact value, but for the linear approximation to that value. You do **not** need to "simplify" your answer!

b) In what direction will f increase most rapidly at $(5,1,2)$? Write a unit vector in that direction.
You do **not** need to "simplify" your answer!

c) What is the directional derivative of f at $(5,1,2)$ in the direction found in b)?
You do **not** need to "simplify" your answer!

(18) 3. Suppose $\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$ where P, Q, and R are continuously differentiable functions. Suppose C is the surface of the unit cube in $\mathbb{R}^3$ ($0 \le x \le 1$ and $0 \le y \le 1$ and $0 \le z \le 1$), $\mathbf{n}$ is the outward unit normal to C, and D is the interior of the unit cube. Prove that $\iint_C \mathbf{F} \cdot \mathbf{n}\,dS = \iiint_D \operatorname{div} \mathbf{F}(x,y,z)\,dV$.

(20) 4. Find four points in $\mathbb{R}^3$ which have positive and equal distance from each other. Then find the cosine of the angle between two faces of a regular tetrahedron.

(20) 5. Suppose $\mathbf{F} = -2xz\mathbf{i} + y^2\mathbf{k}$. **Note** There is *no* $\mathbf{j}$ component in $\mathbf{F}$.

a) Compute curl $\mathbf{F}$.

b) Compute the outward unit normal $\mathbf{n}$ for the sphere $x^2 + y^2 + z^2 = a^2$.

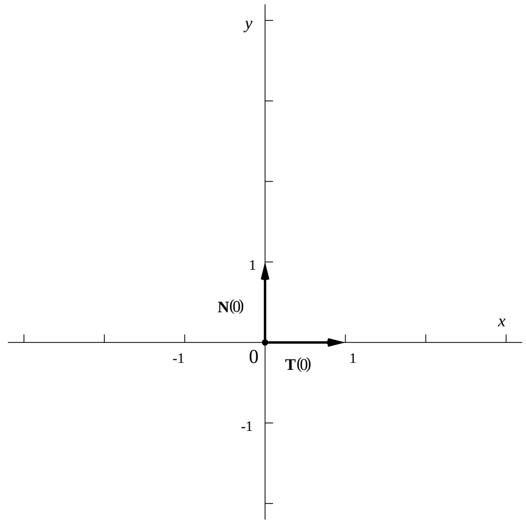
c) If R is any region on the sphere $x^2 + y^2 + z^2 = a^2$, verify that $\iint_R (\text{curl } \mathbf{F}) \cdot \mathbf{n}\,dS = 0$.

d) Suppose C is a simple closed curve on the sphere $x^2 + y^2 + z^2 = a^2$. Show that the line integral $\int_C -2xz\,dx + y^2\,dz = 0$.

Comment Do *not* attempt a direct computation! Use c) and one of the big theorems.

This problem was taken in part from an MIT practice exam.

(16) 6. Suppose that a smooth plane curve is parameterized by arc length s, and that for $0 \le s \le 10$, the curvature as a function of arc length is given by $\kappa(s) = 2^{-\frac{s}{10}}$. Suppose that when $s = 0$, the position on the curve is $(0,0)$ and the value of $\mathbf{T}$ is $\mathbf{i}$ and the value of $\mathbf{N}$ is $\mathbf{j}$, as shown.



a) Sketch the piece of the curve for s in the interval $[0,10]$ as well as you can using all the information given.

b) Answer the following question with a brief explanation: could the curve possibly go through the point $(0,1)$?

(20) 7. Suppose $f(x,y,z) = xy^2z^3$.

a) Compute $\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx$.

b) Write the integral in a) as a sum of one or more iterated integrals in $dx\,dy\,dz$ order. You are *not* asked to integrate your answer, only to set it up.

(16) 8. The average value of a function f defined in a region R of $\mathbb{R}^3$ is $\frac{\iiint_R f\,dV}{\iiint_R dV}$. Compute the average distance to the center of a sphere of radius a.

(16) 9. Suppose $f(a,b,c)$ is a twice differentiable function of three variables, and $g(x,y)$ is defined by $g(x,y) = f(y^2, xy, -x^2)$. Suppose that $D_1f(1,1,-1) = A$, $D_2f(1,1,-1) = B$, $D_3f(1,1,-1) = C$, $D_1D_1f(1,1,-1) = D$, $D_1D_2f(1,1,-1) = E$, $D_1D_3f(1,1,-1) = F$, $D_2D_2f(1,1,-1) = G$, $D_2D_3f(1,1,-1) = H$, and $D_3D_3f(1,1,-1) = I$ where D_j denotes the partial derivative with respect to the j^{th} variable.

Compute $\frac{\partial^2 g}{\partial x \partial y}(1,1)$ in terms of A, B, C, D, E, F, G, H, and I.

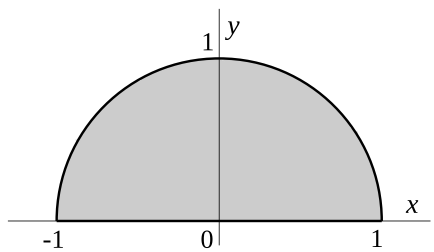
Hint I believe the answer needs exactly five of these.

This problem was taken in part from *Calculus A Complete Course* by Robert A. Adams.

(16) 10. Compute the double integral of

$$f(x,y) = y^2\left(1 + \sin(x^3)\right) + x^2 + y$$

over the upper half of the interior of the unit circle, where $y \ge 0$.



(20) 11. Suppose a vector field is defined by $\mathbf{F}(x,y,z) = \left(3x^2y + yz\cos(xz) + 2\right)\mathbf{i} + \left(x^3 + \sin(xz)\right)\mathbf{j} + \left(xy\cos(xz)\right)\mathbf{k}$.

a) Determine whether there is a scalar function $P(x,y,z)$ defined everywhere in space such that $\nabla P = \mathbf{F}$. If there is such a P, find it; if there is not, explain why not.

b) Compute the integral $\int_W \mathbf{F}\cdot\mathbf{T}\,ds$, where W is any curve STARTing from $(1,1,\frac{\pi}{2})$ and ENDing at $(2,2,\pi)$.

Use information gotten from your answer to a) to help.

Final Exam for Math 291, section 1

May 9, 2003

NAME ____

Do all problems, in any order.
Show your work. An answer alone may not receive full credit.
You may use a calculator only during the last 20 minutes.
No notes may be used on this exam. A page with formulas will be supplied.

Problem Number	Possible Points	Points Earned:
1	20	
2	18	
3	18	
4	20	
5	20	
6	16	
7	20	
8	16	
9	16	
10	16	
11	20	
Total Points Earned:		

Leave answers in "unsimplified" form: $15^2 + (.07) \cdot (93.7)$ **is preferred to** 231.559. **You should know simple exact values of transcendental functions such as** $\cos\left(\frac{\pi}{2}\right)$ **and** $\exp(0)$. **Traditional constants such as** π **and** e **should be left "as is" and not approximated.**