

Formula sheet for the first exam in Math 291, spring 2003

FIRST VERSION 3/2/2003: CORRECTED FROM FALL 2003; AMENDED 3/13/2003

Cauchy-Schwarz: $|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$. Triangle inequality: $\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$.

Distance from $P_0(x_0, y_0, z_0)$ to $P_1(x_1, y_1, z_1)$ is $\sqrt{(x_0 - x_1)^2 + (y_0 - y_1)^2 + (z_0 - z_1)^2}$.

Distance from $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz = d$ is $\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Sphere: $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$

Plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ where $\mathbf{n} = \langle a, b, c \rangle$

Line: $\{x = x_0 + at, y = y_0 + bt, z = z_0 + ct\}$ through (x_0, y_0, z_0) in direction $\langle a, b, c \rangle$

$\|\mathbf{a}\| = \sqrt{(a_1)^2 + (a_2)^2 + (a_3)^2}$

$|\mathbf{a} \cdot \mathbf{b}| = \|\mathbf{a}\| \|\mathbf{b}\| \cos \theta$ (If = 0, then $\mathbf{a} \perp \mathbf{b}$.) $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$ (If $\mathbf{a} \parallel \mathbf{b}$, this = 0.)

$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

$\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|}$ $\text{proj}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$

Volume of a parallelepiped with edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$: $\|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\|$

Arc length: $\int_a^b \|\mathbf{r}'(t)\| dt$ $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$ $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$ $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$

$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} \stackrel{2 \text{ dim}}{=} \frac{|y''(t)x'(t) - x''(t)y'(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}} \stackrel{y=f(x)}{=} \frac{|f''(x)|}{(1 + (f'(x))^2)^{3/2}}$

$\tau = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|^2}$. Frenet-Serret: $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$, $\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$, $\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$.

Tangent plane to $z = f(x, y)$ at $P(x_0, y_0, z_0)$: $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Linear approximation to $f(x, y)$ at (a, b) : $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

Tangent plane to $F(x, y, z) = 0$:

$$F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

If y implicitly defined by $y = f(x)$ in $F(x, y) = 0$ then $\frac{dy}{dx} = -\frac{F_x}{F_y}$.

If z implicitly defined by $z = f(x, y)$ in $F(x, y, z) = 0$ then $z_x = -\frac{F_x}{F_z}$ and $z_y = -\frac{F_y}{F_z}$.

$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$ $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

Some chain rules:

If $z = f(x, y)$ and $x = x(t)$ and $y = y(t)$, then $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

If $z = f(x, y)$ and $x = g(s, t)$ and $y = h(s, t)$, then $\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial g}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial h}{\partial s}$.

Suppose $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let $H = H(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

a) If $H > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

b) If $H > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

c) If $H < 0$, then $f(a, b)$ is not a local maximum or minimum (f has a saddle point).

A real-valued function $F(\mathbf{x})$ is continuous at \mathbf{x}_0 if, given any $\varepsilon > 0$, there is a $\delta > 0$ so that whenever $\|\mathbf{x} - \mathbf{x}_0\| < \delta$, then $|F(\mathbf{x}) - F(\mathbf{x}_0)| < \varepsilon$.