291:01 **Two Lagrange multiplier problems** 3/12/2002

Problem #1

The unit ball, occupying points (x, y, z) in $\mathbb{R}^3$ with $F(x, y, z) = x^2 + y^2 + z^2 \le 1$, has a temperature distribution given by $G(x, y, z) = xy^2z^3$. What are the hottest and coldest temperatures on the unit ball, and where do they occur?

• Interior extreme values

Since $\nabla G = y^2z^3\mathbf{i} + 2xyz^3\mathbf{j} + 3xy^2z^2\mathbf{k}$, any local max/min must occur where either $y = 0$ or $z = 0$ or both are 0. So this function has *an infinite number of critical points* <u>but</u> none of these points can be absolute max's or min's. This is because G's value on any critical point is 0, and G has both positive and negative values on the ball. The level set associated with 0 (those (x, y, z)'s with $G(x, y, z) = 0$) is very large, and includes all the critical points. This level set can be disregarded as we look for absolute max and min.

• Boundary extreme values

We look for (x, y, z) on the boundary, $x^2 + y^2 + z^2 = 1$, satisfying $\nabla F = \lambda \nabla G$. This vector equation is a system of three scalar equations $\begin{cases} 2x = \lambda y^2z^3 \\ 2y = \lambda 2xyz^3 \\ 2z = \lambda 3xy^2z^2 \end{cases}$. We can exclude having *any* variable vanish (that's covered in the previous discussion). Divide the first equation by the second and get $\frac{x}{y} = \frac{y}{2x}$ so $2x^2 = y^2$. Divide the first equation by the third and get $\frac{x}{z} = \frac{z}{3x}$ so $3x^2 = z^2$. Therefore $6x^2 = 1$ and $x = \pm\frac{1}{\sqrt{6}}$ and $y = \pm\frac{\sqrt{2}}{\sqrt{6}}$ and $z = \pm\frac{\sqrt{3}}{\sqrt{6}}$ at any extreme point on the boundary. There are 8 possible points since the signs are unrelated. The product of the signs determines whether each is one of the 4 points with the highest temperature (+ sign, temperature $= \frac{\sqrt{3}}{36}$) or one of the 4 points with lowest temperature ($-$ sign, temperature $= -\frac{\sqrt{3}}{36}$).

• Comments, pictures, and summary

This problem has a great deal of symmetry which makes it easier to solve. The **first picture** shown is the constraint and was obtained from this command:

```
A:=implicitplot3d(x^2+y^2+z^2=1,x=-1.5..1.5,y=-1.5..1.5,z=-1.5..1.5,
color=green,axes=normal,scaling=constrained,grid=[20,20,20]);
```

The **second picture** shows the level set where $G = 0$, which contains all the critical points. The **third picture** shows the level set where $G = -.05$, a "typical" level set with 4 pieces, each a surface "swooping in" towards $(0, 0, 0)$ and then asymptotic towards the coordinate planes. These 4 pieces occur in the octants where the product of the coordinate signs is negative. The **fourth picture** shows the constraint and the level set of the objective function corresponding to the minimum, $-\frac{\sqrt{3}}{36}$. Each piece of the level set is tangent to the constraint.

I defined a function in `Maple` to draw these graphs.

```
SURF:=v->implicitplot3d(x*y^2*z^3=v,x=-2.5..2.5,y=-2.5..2.5,z=-2.5..2.5,
color=pink,axes=normal,scaling=constrained,grid=[20,20,20]);
```

For example the last picture was produced using `display3d(A,SURF(-sqrt(2)/36));`

Problem #2

The unit ball, occupying points (x, y, z) in $\mathbb{R}^3$ with $F(x, y, z) = x^2 + y^2 + z^2 \le 1$, has a temperature distribution given by $H(x, y, z) = x + y^2 + z^3$. What are the hottest and coldest temperatures on the unit ball, and where do they occur?

• **Interior extreme values**

Since $\nabla H = 1\mathbf{i} + 2y\mathbf{j} + 3z^2\mathbf{k}$, we see immediately that H has *no* critical points! So any max or min temperature must be found on the boundary.

• **Boundary extreme values**

Again search for (x, y, z) so that $x^2 + y^2 + z^2 = 1$ and also $\nabla F = \lambda \nabla H$. The scalar system is $\begin{cases} 2x = \lambda \\ 2y = \lambda 2y \\ 2z = \lambda 3z^2 \end{cases}$. The second equation implies either $y = 0$ or $\lambda = 1$. Let's try the second alternative.

<u>If $\lambda = 1$</u>, there is no restriction on y in the second equation, but $x = \frac{1}{2}$.

<u>If $z = 0$</u>, the third equation is satisfied, but $x^2 + y^2 + z^2 = 1$ gives $y = \pm\sqrt{1 - \frac{1}{4}}$.

So we have these candidates: $\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}, 0\right)$. These points have $H = \frac{1}{2} + \frac{3}{4} = \frac{5}{4} = \mathbf{1.25}$.

<u>If $z \neq 0$</u>, the third equation gives $z = \frac{2}{3}$, and then $y = \pm\sqrt{1 - \frac{1}{4} - \frac{4}{9}} = \pm\sqrt{\frac{11}{36}}$. Now these candidates appear: $\left(\frac{1}{2}, \pm\frac{\sqrt{11}}{6}, \frac{2}{3}\right)$. These two points have $H = \frac{119}{108} \approx \mathbf{1.10185}$.

<u>If $\lambda \neq 1$</u>, then $y = 0$ certainly. λ cannot be 0 for then both of x and z are 0 and (x, y, z) can't be on the unit sphere. If λ is not 0, then x is not 0. Either z is 0 or not 0.

<u>If $z = 0$</u>, then $x = \pm 1$ and we have candidates $(\pm 1, 0, 0)$ with $H = \pm\mathbf{1}$.

<u>If $z \neq 0$</u>, then $\lambda z = \frac{2}{3}$ and $xz = \frac{1}{3}$, so that the constraint equation $x^2 + y^2 + z^2 = 1$ becomes $9x^4 + 9x^2 - 1 = 0$. The roots of this equation are (really!) $\frac{\pm\sqrt{15}\pm\sqrt{3}}{6}$ which are approximately $\pm.35682$ and $\pm.93417$. We get 4 more candidates: $\pm(.35682, 0, .93417)$ and $\pm(.93417, 0, .35682)$. The $\pm$'s are outside the parentheses because the signs in the triples must be selected together. The H values on the first triples are $\pm\mathbf{.97960}$ and on the second, $\pm\mathbf{1.17205}$.

• **Comments, pictures, and summary**

The constraint is the same as in the first problem. The **first picture** shown here is a collection of three level sets (corresponding to $H = 6$ [darkest], $H = .5$, and $H = -5$ [lightest]). The **second picture** shows the constraint surface together with the level set corresponding to the minimum temperature, -1.17205, which is reached at the point $(-.93417, 0, -.35682)$: the ball is tangent to that level surface. The **third picture** shows the constraint together with the level surface corresponding to the maximum temperature, 1.25, which is actually achieved at two points. This situation may be more difficult to understand. The constraint surface is "inside" the level surface, and is tangent at two points. The **last picture** is perhaps the most mysterious. I am trying to show what happens at one of the intermediate values picked out by the multiplier method, the value 1.10185. This value is attained at two points on the constraint surface, the sphere. The picture "zooms in" at the point in the first octant. The lighter color is the level set of the objective function, and the darker color is the sphere. You can't see much beyond the local structure of the picture, but it should indicate to you that near the point identified, $\left(\frac{1}{2}, \frac{\sqrt{11}}{6}, \frac{2}{3}\right)$, there is sort of a "saddle" effect: temperatures on the surface of the sphere are both higher and lower. Away from local extrema, the pictures may be *very* complex.

This example still has a great amount of symmetry. Investigating it and producing these pictures took some time and effort. The pictures in Postscript take 6.4 megabytes of storage. There are colored pictures on the web (probably in the course diary file). In GIF format, the pictures use about a tenth of a megabyte of storage.