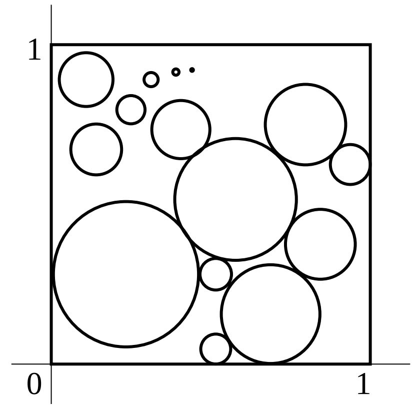
Disconcerting problems about dimensions

My principal aim in writing these problems is to convince you that geometry and calculus in dimensions greater than 1 can be *different* from dimension 1. I also want to see how you can write mathematics. While you may consult with others and with me about the solution of these problems, I would like each student to write answers independently. One of my goals in Math 291 is to improve your written and oral ability to explain mathematics.

Discussion and statement of the first problem

A *sequence of bubbles* is an infinite sequence of circles in the unit square of the plane, $[0,1] \times [0,1]$, whose interiors do not overlap. The center and radius of each circle should be specified in some algebraic or geometric fashion. A picture of some bubbles in one sequence appears to the right.



The problem

Is there a sequence of bubbles so that

i the sum of the bubble areas is finite and

ii the sum of the bubble circumferences is infinite?

What you should do

Either give an example of such a sequence of bubbles as explicitly as you can, or explain why no example exists. Your response should contain a discussion supporting your assertion written in complete English sentences.

Discussion and statement of the second problem

We begin with some terminology and notation.

- $\mathbb{R}^n$ (pronounced "are en") is n-dimensional Euclidean space. A point p in $\mathbb{R}^n$ is an n-tuple of real numbers: $p = (x_1, x_2, \ldots, x_n)$. The numbers x_j are called the coordinates of p. For example, $(1, 2, -3.8, 400, 5\pi)$ is a point in $\mathbb{R}^5$.
- If $p = (x_1, x_2, \ldots, x_n)$ and $q = (y_1, y_2, \ldots, y_n)$ are two points in $\mathbb{R}^n$, the distance from p to q is defined to be $D(p,q) = \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2}$. This is supposed to be a natural generalization of the usual formulas for distance in $\mathbb{R}^2$ and $\mathbb{R}^3$: a repetition n times of the Pythagorean formula. For example, $p = (1, 7, 8, -4)$ and $q = (2, -3, 9, 9)$ are points in $\mathbb{R}^4$, and the distance between them is $\sqrt{(1-2)^2 + (7-(-3))^2 + (8-9)^2 + (-4-9)^2} = \sqrt{271} \approx 16.46208$. The formula for $D(p,q)$ satisfies the usual rules for distances. The text uses $|pq|$ to denote the distance from p to q.
- The origin in $\mathbb{R}^n$ is $\mathbf{0} = (0, 0, \ldots, 0)$, the n-tuple which is all 0's.
- The n-dimensional unit cube is the collection of points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ satisfying all of these inequalities: $0 \le x_j \le 1$ for $1 \le j \le n$.
- The corners of the n-dimensional unit cube are the points $(x_1, x_2, \ldots, x_n)$ where each x_j is either 0 or 1. Each of the n choices of the coordinates for a corner can be made independently and there are two alternatives for each coordinate. Therefore the n-dimensional unit cube has 2^n corners.

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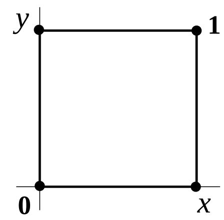
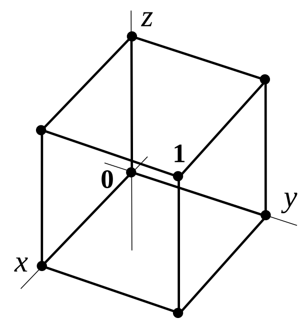
Here are some familiar unit cubes, in 2 and 3 dimensions. The corners are marked with •'s. The 2-dimensional cube has $2^2 = 4$ corners. The 3-dimensional cube has $2^3 = 8$ corners.





Do the following exercises before starting the problem. Please *don't* hand in solutions! You may ask me about them. Answers ("spoilers") without explanation appear at the bottom of the page. I suggest you look at them *after* you try the problems.

Exercise 1 Suppose $\mathbf{1} = (1, 1, \ldots, 1)$, the n-tuple which is all 1's. Compute the distance between $\mathbf{0}$ and $\mathbf{1}$, which are both corners of the n-dimensional cube. This should convince you that at least *part* of the n-dimensional cube "sticks out" far away from the origin.

Exercise 2 The 20-dimensional unit cube has $2^{20} = 1{,}048{,}576$ corners, far too many to list explicitly. You may need to use a calculator to answer the questions below.

a) How many corners of the 20-dimensional cube have *all* 0's in their coordinates? How many have *exactly one* 1 in their coordinates? How many have *exactly two* 1's in their coordinates? How many have *exactly three* 1's in their coordinates? How many have *exactly four* 1's in their coordinates? [This starts out very easy, then becomes harder.]

b) Use a)'s answer to find the total number of corners of the 20-dimensional unit cube which have 1's in at most four coordinates.

c) Use b)'s answer to find the total number of corners of the 20-dimensional unit cube whose distance to $\mathbf{0}$ is at most 2. ($2 = \sqrt{1^2 + 1^2 + 1^2 + 1^2}$.)

d) Use c)'s answer to find the proportion of the corners of the 20-dimensional unit cube which have distance to the origin greater than 2.

Unit cubes are quite weird when n is large.

The problem

Suppose A is a positive constant and $\#(n, A)$ is the number of corners of the n-dimensional unit cube whose distance to $\mathbf{0}$ is greater than A. Then $\displaystyle\lim_{n\to\infty} \frac{\#(n,A)}{2^n} = 1$: "almost all" of the corners of the cube are eventually, as dimension grows, farther away from $\mathbf{0}$ than A.

What you should do

Verify the limit statement above. You will need facts from calculus (quote them) about the asymptotic growth of polynomials compared to exponentials. Your response should contain a discussion supporting your assertion written in complete English sentences.

Hints

Begin with $A = 2$: in exercise 2, generalize to $\mathbb{R}^n$ in place of $\mathbb{R}^{20}$. The limit for $A = 2$ compares the growth of a fourth degree polynomial with that of an exponential function. Then consider $A = 78$. The polynomial's degree is now 78^2 but the asymptotics (polynomial growth versus exponential growth) remain qualitatively the same. What about $A = 78.5$? Please hand in only a report on the general case, if possible.

Remarks on notation $[\![x]\!]$ means the "integer part" of x (see p. 108 of Stewart), so $[\![78.5]\!] = 78$. The binomial coefficients $\binom{a}{b}$ (for a and b integers with $a \geq b$) are defined on p. 762. You may know or should learn that $\binom{a}{b}$ is the number of ways to choose b objects from a objects. Some links to web pages explaining this are on the course web page.

Answers to exercises: 1. $\sqrt{n}$ | 2. a) 1, 20, 190, 1140, 4845 | 2. b) 6196 | 2. c) 6196 again. | 2. d) $\approx$.99409