Please write solutions to two of these problems. Hand them in Monday, March 24. The written solutions should be accompanied by explanations using complete English sentences. Students should work **alone**. *They may ask me questions.*

1. a) What is the maximum value of the function $f(x, y) = 3x+5y$ subject to the constraint $x^2+y^2 = 1$, and where is it attained? Draw a picture of the constraint and the appropriate level set of the objective function.

b) Suppose n is a positive real number. What is the maximum value of the function $f(x, y) = 3x + 5y$ subject to the constraint $x^n + y^n = 1$ and where is it attained? Your answers should all be functions of n.

c) What happens to the maximum value found in b) when $n \to \infty$? Try to draw a picture of the situation when n is large.

d) What happens to the maximum value found in b) when $n \to 0^+$? Try to draw a picture of the situation when n is small.

Comment Graphing programs don't seem to handle the extreme situations described in c) and d) very well. You may need to think about what appropriate pictures should be.

2. Doreen and Charlemagne want to build a box with a square base and an open top which will hold 5 ft^3. The bottom and one side are to be built of material which is *twice* as expensive as ... Hold it, hold it! Forget that, and just find the maximum and minimum of $Axy+Byz+Cxz$ subject to the constraint $xyz = 1$ where A, B, and C are all positive.
Comment Be careful. The problem may not be *nice*.

3. `Real life!` Find the maximum and minimum of $8x + 7y$ subject to the restrictions: $y - (.9)x \leq .5$ <u>and</u> $y - (1.2)x \geq .3$ <u>and</u> $y + (2.3)x \geq .7$ <u>and</u> $y - (2.1)x \geq .1$.[α]

4. Polynomials have roots. For example, $(x - 1)(x + 2)(x - 3) = x^3 - 2x^2 - 5x + 6$ which means that $x^3 - 2x^2 - 5x+6$ is 0 when $x = 1$ or $x = -2$ or $x = 3$. Low degree polynomials have algebraic recipes for roots in terms of their coefficients. There are no such formulas for higher degree polynomials. If r_1, r_2, and r_3 are the roots of a third degree polynomial, $x^3 + Ax^2 + Bx + C$, then the coefficients (A, B, and C) are functions of the roots (r_1, r_2, and r_3).

a) What are the functions? Verify for your answers that if $\begin{cases} r_1 = 1 \\ r_2 = -2 \\ r_3 = 3 \end{cases}$ then $\begin{cases} A = -2 \\ B = -5 \\ C = 6 \end{cases}$.

b) Suppose the roots are changed: $\begin{cases} r_1 : 1 \to 1.02 \\ r_2 : -2 \to -2.04 \\ r_3 : 3 \to 2.95 \end{cases}$. Predict the *approximate* changes in the coefficients. (Use partial derivatives to get the linearized perturbation; this is easy.)

[α] You should probably begin this problem by drawing a picture of the region. Can there be "interior" extrema" Can there be extreme "interior" to the edges? If you think this through, the problem will reduce to simple computations and comparisons. This is an example of a "linear programming" problem: a simple (in some sense) function to be extremized and a domain whose boundary is made up of flat simple pieces. It is "easy", but real-life versions may have ten or twenty thousand variables and much more than that many inequalities describing the domain of the function. It is then not "easy" to describe how to extremize the function subject to the given constraints, even with extensive computer help.

c) Suppose now that the coefficients are changed. That is, we observe new coefficients: $\begin{cases} A = -2.03 \\ B = -5.02 \\ C = 6.01 \end{cases}$. Approximate the roots which would give these coefficients. (This is harder, and you need a *new idea*: what perturbations in the roots will, to first order, give these perturbations in the coefficients? You must solve a "system" of three linear equations in three unknowns.)

5. It is certainly possible for the set of critical points of a function defined in $\mathbb{R}^3$ to be a point (e.g., $x^2 + y^2 + z^2$) or a line (e.g., $x^2 + y^2$) or a plane (e.g., x^2). Can you create a function $F : \mathbb{R}^3 \to \mathbb{R}$ whose set of critical points is the twisted cubic, $\mathbf{c}(t) = (t, t^2, t^3)$?

6. a) **The 1-dimensional wave equation** Suppose that $H(x,t)$ represents the height of a vibrating string over the point x of the real line at time t. Then for small vibrations and for homogeneous strings (think of a guitar string) H satisfies the partial differential equation $\frac{\partial^2 H}{\partial x^2} - \frac{\partial^2 H}{\partial t^2} = 0$. Verify that if $f(w)$ is any twice-differentiable function of one variable, then $H(x,t) = f(x-t)$ satisfies the wave equation. Comment on how the wave shape (the graph of f) travels along the string as t changes. If H_1 and H_2 satisfy the wave equation, verify that $H_1 + H_2$ and cH_1 (where c is any constant) also satisfy the wave equation. This is called "the principle of superposition".

b) **The Korteweg-de Vries equation** The following paragraph was written in 1844 by John Scott Russell, a Scottish engineer.

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.

Quoted on `http://www.ma.hw.ac.uk/~chris/scott_russell.html`

This was the first published record of what is now called a soliton[β]. In 1895 the phenomenon was described with a partial differential equation named the *Korteweg-de Vries equation* (KdV): $u_t + u_{xxx} + 6uu_x = 0$ Korteweg and de Vries stated the equation with supporting reasoning. This equation is non-linear and does not satisfy the "principle of superposition". KdV and related equations have turned out to be very important both theoretically and in practical applications (fiber optics, transmission of nerve impulses, some chemical reactions).

i) If $K > 0$, show that $u_K = \frac{K}{2}\left(\operatorname{sech}\left(\frac{\sqrt{K}}{2}(x - Kt)\right)\right)^2$ is a solution of KdV[γ].

ii) Verify that $7u_1$ and $u_2 + u_3$ are *not* solutions of KdV.

iii) Describe a connection between the speed of the soliton and its maximum height.[δ]

More information `http://math.cofc.edu/faculty/kasman/SOLITONPICS/index.html`

β Google lists more than 111,000 web pages in response to "soliton".

γ Yes: sech means *hyperbolic secant*

δ Russell wrote (1885), "The sound of a cannon travels faster than the command to fire it."