640:311:01 **Answers to the First Exam** **3/7/2003**

The answers to part 1 were provided in part 2.

(14) 6. Suppose the sequence (x_n) is defined by $x_n := \dfrac{n-1}{5n+7}$. Find x so that (x_n) converges to x. Prove your assertion using the definition of convergence.

Answer I claim that $x = \dfrac{1}{5}$. Suppose $\varepsilon > 0$ is given. By the Archimedean Property, there is an integer $K \in \mathbb{N}$ so that $K > \dfrac{\frac{12}{5\varepsilon} - 7}{5}$. Then if $n \geq K$, $n > \dfrac{\frac{12}{5\varepsilon} - 7}{5}$ so that $5n > \dfrac{12}{5\varepsilon} - 7$ and then $5n + 7 > \dfrac{12}{5\varepsilon}$. Further, we have $5\varepsilon > \dfrac{12}{5n+7}$ so that $\varepsilon > \dfrac{12}{5(5n+7)}$. But $\dfrac{12}{5(5n+7)} = \left|\dfrac{-12}{5(5n+7)}\right| = \left|\dfrac{n-1}{5n+7} - \dfrac{1}{5}\right|$ and we have verified the definition of convergence. (The reader should recognize that preliminary algebra was done "off the page".)

(12) 7. Suppose that the sequence (x_n) converges to x and the sequence (y_n) converges to y. Prove that the sequence (z_n) defined by $z_n := x_n + y_n$ converges to $x + y$.

Answer Suppose $\varepsilon > 0$ is given. The definition of convergence of a sequence implies that there are $J(\varepsilon)$ and $K(\varepsilon)$ in $\mathbb{N}$ so that if $n \geq J(\varepsilon)$ then $|x_n - x| < \varepsilon$ and if $n \geq K(\varepsilon)$ then $|y_n - y| < \varepsilon$.

Now consider $W = \max\left(J\left(\frac{\varepsilon}{2}\right), K\left(\frac{\varepsilon}{2}\right)\right)$. For $n \geq W$, we know $|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$. The triangle inequality implies that $|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \overset{\triangle}{\leq} |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ and we have verified the definition of convergence.

(14) 8. Suppose S is a nonempty subset of $\mathbb{R}$ which is bounded above, and $a \in \mathbb{R}$. Define a subset T of $\mathbb{R}$ by $T := \{x : \exists s \in S \text{ so that } x = a + s\}$. ($T$ is S "translated by a".) Prove that T is bounded above, and prove that $\sup T = a + \sup S$.

Answer Since S is bounded above and nonempty, the Completeness Axiom implies that S has a least upper bound, w. Since w is an upper bound of S, if $s \in S$, $w \geq s$. Then $w + a \geq s + a$ for all $s \in S$, so that if $x \in T$, $w + a \geq x$. Therefore T is bounded above, and $w + a$ is an upper bound of T. Also T is nonempty since S is nonempty. By the Completeness Axiom, T also has a least upper bound, v. Since $w + a$ is an upper bound of T, $w + a \geq v$. Suppose $w + a > v$. Then $w > v - a$. By the criterion for Least Upper Bound, there is $s \in S$ with $w \geq s > v - a$. Then $s + a > v$, and this is a contradiction, since $s + a$ is an element of T and v is supposed to be an upper bound of T. Therefore $\sup T = a + \sup S$.

Comment We could also show that $v \geq w + a$ in the following way: $v - a$ must be an upper bound of S. If not, there is an element s of S with $s > v - a$ which implies $a + s > v$, contradicting v being an upper bound of T. Since $v - a$ is therefore an upper bound of S, $v - a \geq w$ and $v \geq w + a$.

(12) 9. Prove that $2n^2 < 3^n$ for all $n \in \mathbb{N}$.

Comment You may need to verify more than one example numerically.

Answer Let $\mathcal{P}(n)$ be the statement $2n^2 < 3^n$ for $n \in \mathbb{N}$. I will prove that $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$ by Mathematical Induction. $\mathcal{P}(1)$ is $2(1^2) < 3^1$ which is true since $2 < 3$.

Now suppose $\mathcal{P}(n)$ is true for some $n \in \mathbb{N}$: $2n^2 < 3^n$. We know $n+1 = \left(\frac{n+1}{n}\right) n$. We can divide by n since n is positive. $\frac{n+1}{n} = 1 + \frac{1}{n} \leq 2$ if $n \geq 1$, and therefore $\left(\frac{n+1}{n}\right)^2 \leq 4$. Our simple approach to verify $\mathcal{P}(n+1)$ would be to multiply both sides of $2n^2 < 3^n$ by $\left(\frac{n+1}{n}\right)^2$ but since the overestimate we have only provides 4 we wouldn't get 3^{n+1}.

Let us verify $\mathcal{P}(2)$: $2n^2 = 2(2)^2 = 8$ and $3^2 = 9$, and since $9 > 8$, $\mathcal{P}(2)$ is true. Now again suppose $\mathcal{P}(n)$ is true for some $n \in \mathbb{N}$ with $n \geq 2$. Here $\frac{n+1}{n} = 1 + \frac{1}{n} \leq \frac{3}{2}$, so $\left(\frac{n+1}{n}\right)^2 \leq \left(\frac{3}{2}\right)^2 = \frac{9}{4} < 3$. If we multiply $2n^2 < 3^n$ by $\left(\frac{n+1}{n}\right)^2 < 3$ we get $2(n+1)^2 < 3^{n+1}$, which is the statement $\mathcal{P}(n+1)$. So the inductive step is verified, and the proof is completed.

(14) 10. Suppose that S is a nonempty subset of $\mathbb{R}$ with the property that if $a \in S$ then $a^2 \in S$. Prove that if S is bounded above, then $\sup S \leq 1$.

Answer Since S is bounded above and nonempty, the Completeness Axiom applies, and $w = \sup S$ exists. If $w > 1$, then $\sqrt{w} > 1$ also, and we know that $w > \sqrt{w}$. The criterion for least upper bound implies that there is $a \in S$ with $w \geq a > \sqrt{w}$. But since $a \in S$, $a^2 \in S$, and $a^2 > (\sqrt{w})^2 = w$, which contradicts w being an upper bound of S. Therefore the assumption $w > 1$ is false.

(14) 11. Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that $|x_n - y_n| < \varepsilon$ for all $n \geq M(\varepsilon)$. Does it follow that (y_n) is convergent? Prove your assertion.

Answer Yes, (y_n) is convergent, and it converges to the same limit as the sequence (x_n). Suppose x is the limit of the sequence (x_n). Then we know from the definition of convergence that if $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ so that for $n \geq K(\varepsilon)$, $|x_n - x| < \varepsilon$. Now consider $W = \max\left(K\left(\frac{\varepsilon}{2}\right), M\left(\frac{\varepsilon}{2}\right)\right)$. For $n \geq W$, $|y_n - x| = |(y_n - x_n) + (x_n - x)| \overset{\triangle}{\leq} |y_n - x_n| + |x_n - x|$ using the triangle inequality. The expression $|y_n - x_n|$ is less than $\frac{\varepsilon}{2}$ by the condition on n and the problem statement. The second expression, $|x_n - x|$, is less than $\frac{\varepsilon}{2}$ again by the condition on n involving convergence of the sequence (x_n). Therefore $|y_n - x| < \varepsilon$ for $n \geq W$, and we have verified that (y_n) converges to x. (This is problem 21 from section 3.2 of the text.)