**640:311:01** **Answers to the Second Exam** **4/18/2003**

The answers to part 1 were provided in part 2.

(14) 6. Suppose that $I = [a, b]$ is a closed, bounded interval, and $f: I \to \mathbb{R}$ is a function with $f(x) > 0$ for all $x \in I$. Let $z = \inf\{f(x) : x \in I\}$.
a) If $f$ is continuous on all of $[a, b]$, prove that $z > 0$.

**Answer** Since $f$ is continuous on $[a, b]$, the Extreme Value Theorem asserts that there is $v \in [a, b]$ so that $f(v) \le f(x)$ for all $x \in [a, b]$. Therefore $f(v)$ is a lower bound and it is positive. Since $z = f(v)$, $z$ must be positive.

b) Give an example showing that if $f$ is *not* continuous but still has the property that $f(x) > 0$ for all $x \in I$, $z$ may be 0.

**Answer** Suppose $[a, b] = [0, 1]$, If $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$ then $f$ is not continuous at 0. Also $z = 0$ since $\inf\{(0, 1]\} = 0$. Here $f(x)$ is never 0 for all $x \in [0, 1]$.

(12) 7. Find a positive number $b$ so that if $|x - 1| < b$ then $|x^3 - 1| < \frac{1}{100}$. Prove your assertion.

**Answer** Algebra tells us that $x^3 - 1 = (x - 1)(x^2 + x + 1)$. If $|x - 1| < 1$, then $0 < x < 2$ so that $1 < x^2 + x + 1 < 7$. We can take $b = \frac{1}{700}$. Since this $b$ is less than 1, we know $|x^2 + x + 1| < 7$. Then we see that $|x^3 - 1| = |x - 1| \cdot |x^2 + x + 1| < \frac{1}{700} \cdot 7 = \frac{1}{100}$. Certainly there are other valid answers.

(12) 8. Suppose the set $A$ is all negative rational numbers and the number 1. That is, $A = \{x \in \mathbb{R} : x \text{ is rational and } x < 0\} \cup \{1\}$. Find all the cluster points of $A$. Explain briefly why each point so identified *is* a cluster point, and explain briefly why each point excluded is *not* a cluster point.

**Answer** Let $C$ be the set of cluster points of $A$. Then $C = (-\infty, 0]$. If $x \le 0$ and $\delta > 0$, the interval $(x - \delta, x)$ is open and non-empty, and therefore by the density of $\mathbb{Q}$ there must be a rational number $r \in (x - \delta, x)$. Since $x \le 0$, $r$ must be negative and therefore in $A$. So every element of $C$ is a cluster point of $A$. We must show that positive numbers are not cluster points of $A$. 1 is not a cluster point, since there are no elements of $A$ in the set described by the inequalities $0 < |x - 1| < 1$ corresponding to $\delta = 1$. If $w > 0$ and $w \ne 1$, then take $\delta = \min(|w - 1|, w)$. Both of those numbers are positive so $\delta$ is positive. The set of $x$'s satisfying $0 < |x - w| < \delta$ contains only positive numbers (because of the $w$ in the specification of $\delta$) and does not contain 1 (because of the $|w - 1|$ in the specification of $\delta$). Therefore no element of $A$ is in that set, so $w$ cannot be a cluster point of $A$.

(12) 9. If $\sum_{j=1}^{\infty} |a_j|$ converges, prove that $\sum_{j=1}^{\infty} a_j$ converges.

**Hint** The Cauchy criterion and the $\overset{\triangle}{\le}$ and the Cauchy criterion.

**Answer** Let $x_n = \sum_{j=1}^{n} |a_j|$ and let $y_n = \sum_{j=1}^{n} a_j$. Since $(x_n)$ converges, it satisfies the Cauchy criterion: given $\varepsilon > 0$, there is $K(\varepsilon) \in \mathbb{N}$ so that for $m > n \ge K(\varepsilon)$, $|x_m - x_n| < \varepsilon$. But $|y_m - y_n| = \left| \sum_{j=n+1}^{m} a_j \right| \overset{\triangle}{\le} \sum_{j=n+1}^{m} |a_j| = |x_m - x_n|$ so that the sequence $(y_n)$ also satisfies the

Cauchy criterion with the same $K(\varepsilon)$. So $(y_n)$ must converge. Thus the infinite series $\sum_{j=1}^{\infty} a_j$ converges.

(14) 10. Suppose that a sequence is defined recursively by $\begin{cases} x_1 = 1 \\ x_{n+1} = \sqrt{2x_n + 3} & \text{for } n \in \mathbb{N} \end{cases}$.

Here are approximate values with error < .00001 of the next ten elements of the sequence:
2.23606, 2.73352, 2.90981, 2.96978, 2.98991, 2.99663, 2.99887, 2.99962, 2.99987, 2.99995.

a) Prove that $(x_n)$ is an increasing sequence.

**Answer** We know from what's shown that $x_1 < x_2$. Let $\mathcal{P}_n$ be the inequality $x_n < x_{n+1}$. If we verify that knowing $\mathcal{P}_n$ is true implies that $\mathcal{P}_{n+1}$ is true, we will have proved that the sequence is increasing by mathematical induction. Since $x_n < x_{n+1}$, we know that $2x_n < 2x_{n+1}$ because $2 > 0$. And then $2x_n + 3 < 2x_{n+1} + 3$ by adding 3. And square roots preserve inequalities of numbers (all of these numbers are positive), so that $\sqrt{2x_n + 3} < \sqrt{2x_{n+1} + 3}$. This is exactly $\mathcal{P}_{n+1}$, and we are done.

b) Prove that $(x_n)$ is bounded above.

**Answer** I claim that $x_n \leq 3$ for all $n \in \mathbb{N}$. Let $\mathcal{Q}_n$ be the inequality $x_n \leq 3$. $\mathcal{Q}_1$ is true: $x_1 = 1 < 3$. Assuming $\mathcal{Q}_n$: if $x_n \leq 3$, then $2x_n \leq 6$ and $2x_n + 3 \leq 9$. Taking square roots, we get $\sqrt{2x_n + 3} \leq 3$ which is $\mathcal{Q}_{n+1}$, and we are done.

c) Conclude that $(x_n)$ converges, and find its limit, with brief explanation of your work.

**Answer** $(x_n)$ is a bounded, increasing sequence and therefore converges. If $L$ is the limit of this sequence, then $L^2$ is the limit of $\left((x_n)^2\right)$. But this sequence is $(2x_n + 3)$ which, using our theorems on limits and arithmetic, must converge to $2L + 3$. So $L^2 = 2L + 3$ and $L^2 - 2L - 3 = 0$ so that $(L-3)(L+1) = 0$. $L$ must be either $-1$ or 3. The limit of a non-negative sequence must be non-negative, and therefore $L = 3$.

(16) 11. a) Prove that $F(x) = \frac{1}{x}$ is uniformly continuous on the domain $[1, \infty)$.

**Answer** Of course we link $|F(x) - F(c)|$ and $|x - c|$: $|F(x) - F(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x-c|}{|xc|}$. In this problem both $x$ and $c$ are positive, so $|xc| = xc$. And in part a), $x \geq 1$ and $c \geq 1$ so that $\frac{|x-c|}{xc} \leq |x - c|$. Given $\varepsilon > 0$, take $\delta = \varepsilon$. If $|x - c| < \delta$, $\left|\frac{1}{x} - \frac{1}{c}\right| \leq |x - c| < \delta$, and $|F(x) - F(c)| < \varepsilon$. Therefore $F$ is uniformly continuous on $[1, \infty)$.

b) Prove that $F(x) = \frac{1}{x}$ is not uniformly continuous on the domain $(0, 1]$.

**Answer** Use the algebra above. Let's look at $x_n = \frac{1}{n}$. Then $|x_n - x_{n+1}| = \frac{1}{n(n+1)}$ and this surely converges with limit 0. But $|f(x_n) - f(x_{n+1})| = 1$. Therefore we have created an $\varepsilon$ (the number 1) for which no $\delta$ can be found: the statement "If $|x - c| < \delta$ with $x$ and $c$ both in (0,1], then $|F(x) - F(c)| < 1$" is false for any positive $\delta$ (take $n$ so that $\frac{1}{n(n+1)} \leq \frac{1}{n} < \delta$, possible by the Archimedean Property, and take $x = x_n$ and $c = x_{n+1}$). Of course there are other valid answers to this question.