Math 311:01 # Entrance "exam" answers **1/28/2003**

1. (12 points) Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of real numbers satisfying: $a_1 = 4$ and $a_2 = 5$ and for all integers $n \geq 3$, $a_n = \frac{5a_{n-1}}{6} + \frac{3}{a_{n-2}}$. Prove that for every natural number n, $3 \leq a_n \leq 6$.

Proof We will use mathematical induction. Let S be the subset of $\mathbb{N}$ for which the statement $3 \leq a_n \leq 6$ is true. Certainly S contains 1 and 2, since $3 \leq a_1 = 4 \leq 6$ and $3 \leq a_2 = 5 \leq 6$. We will now show that if $n \geq 3$ and if $n-1$ and $n-2$ are in S, then $n \in S$. This is the inductive step, and will complete the proof that $S = \mathbb{N}$.
Since $n-1 \in S$, $3 \leq a_{n-1} \leq 6$. Multiply this inequality by $\frac{5}{6}$ to obtain $\frac{15}{6} \leq \frac{5a_{n-1}}{6} \leq \frac{30}{6}$. Since $n-2 \in S$, we know that $3 \leq a_{n-2} \leq 6$. All the terms in this expression are positive, so we may take their reciprocals, reverse the inequalities, and obtain a true statement: $\frac{1}{6} \leq \frac{1}{a_{n-2}} \leq \frac{1}{3}$. Multiply this inequality by 3 and obtain $\frac{3}{6} \leq \frac{3}{a_{n-2}} \leq 1$. Now add the inequality $\frac{15}{6} \leq \frac{5a_{n-1}}{6} \leq \frac{30}{6}$ and the inequality $\frac{3}{6} \leq \frac{3}{a_{n-2}} \leq 1$ and get $\frac{18}{6} \leq \frac{5a_{n-1}}{6} + \frac{3}{a_{n-2}} \leq \frac{30}{6} + 1$. Simplify, and note the appearance of a_n: $3 \leq a_n \leq 6$. Therefore $n \in S$ and we are done.

Grading 3 points for using mathematical induction; 3 points for the "initialization" with $n = 1$ and $n = 2$; 6 for the inductive step.

2. (8 points) In the following sentence, assume that the letter f was previously introduced to represent some function from $\mathbb{R}$ to $\mathbb{R}$ and that the letter a was introduced to represent some number.

> There exists a positive real number u such that for every real number x if $|x-a| < u$ then $f(x) \geq f(a)$.

Construct an English sentence that is equivalent to the negation of the above sentence. The sentence you construct should not contain any words of negation such as "no", "not" or "false". Your sentence may use mathematical notation from arithmetic such as "+","−","×","≤", '≥","≠", etc.

Answer Here is one possible correct answer:

> For all positive real numbers u, there is a real number x so that $|x - a| < u$ and $f(x) < f(a)$.

Grading 8 points. 2 points off for each error.

3. (8 points) Prove or disprove the following statement:

> For any three subsets A, B, C of $\mathbb{R}$, if $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ and $A \cap C \neq \emptyset$, then $A \cap B \cap C \neq \emptyset$.

Answer The statement is false. One example displaying this is: $A = \{1, 2\}$ and $B = \{2, 3\}$ and $C = \{3, 1\}$. Then $A \cap B = \{2\} \neq \emptyset$ and $B \cap C = \{3\} \neq \emptyset$ and $A \cap C = \{1\} \neq \emptyset$ while, since A, B, and C have no elements in common, $A \cap B \cap C = \emptyset$.

Grading 8 points: 2 for declaring that the assertion is "false" and 6 for a specific example. Only 2 points for the declaration and then continuing without a specific example or valid explanation.

4. (18 points) Determine whether each of the following sentences is true or false and prove your answers.

a) For all positive real numbers x there exists a positive real number y such that $xy < \frac{1}{100}$.

Answer The statement is true. We may take $y = \frac{1}{1{,}000x}$, valid since $x \neq 0$. This y is positive. Then $xy = \frac{1}{1{,}000}$ is certainly less than $\frac{1}{100}$.

b) There exists a positive real number y such that for all positive real numbers x, $xy < \frac{1}{100}$.

Answer The statement is false. If y is any positive real number, consider the positive real number $x = \frac{1}{y}$. The product xy is 1, which is certainly *not* less than $\frac{1}{100}$.

c) There exists a real number y such that for all positive real numbers x, $xy < 0$.

Answer The statement is true. One such y is -1. For positive x, $(-1)x$ is negative and therefore must be less than 0.

Grading 6 points each: 2 for stating a correct assertion, and then the other 4 for verification of the assertion.