(14) 1. Suppose that $I = [a, b]$ is a closed, bounded interval, and $f\colon I \to \mathbb{R}$ is a function with $f(x) > 0$ for all $x \in I$. Let $z = \inf\{f(x) : x \in I\}$.

a) If f is continuous on all of $[a, b]$, prove that $z > 0$.

b) Give an example showing that if f is *not* continuous but still has the property that $f(x) > 0$ for all $x \in I$, z may be 0.

(12) 2. Find a positive number b so that if $|x - 1| < b$ then $|x^3 - 1| < \frac{1}{100}$. Prove your assertion.

(12) 3. Suppose the set A is all negative rational numbers and the number 1. That is, $A = \{x \in \mathbb{R} : x \text{ is rational and } x < 0\} \cup \{1\}$. Find all the cluster points of A. Explain briefly why each point so identified *is* a cluster point, and explain briefly why each point excluded is *not* a cluster point.

(12) 4. If $\sum_{j=1}^{\infty} |a_j|$ converges, prove that $\sum_{j=1}^{\infty} a_j$ converges.

Hint The Cauchy criterion and the $\overset{\triangle}{\leq}$ and the Cauchy criterion.

(14) 5. Suppose that a sequence is defined recursively by $\begin{cases} x_1 = 1 \\ x_{n+1} = \sqrt{2x_n + 3} & \text{for } n \in \mathbb{N} \end{cases}$.

Here are approximate values with error < .00001 of the next ten elements of the sequence:

2.23606, 2.73352, 2.90981, 2.96978, 2.98991, 2.99663, 2.99887, 2.99962, 2.99987, 2.99995.

a) Prove that (x_n) is an increasing sequence.

b) Prove that (x_n) is bounded above.

c) Conclude that (x_n) converges, and find its limit, with brief explanation of your work.

(16) 6. a) Prove that $F(x) = \dfrac{1}{x}$ is uniformly continuous on the domain $[1, \infty)$.

b) Prove that $F(x) = \dfrac{1}{x}$ is not uniformly continuous on the domain $(0, 1]$.

Answers to Part 1 of Exam 2

1. **The definition of continuity**
Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Then f is continuous at c if, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

2. **A sequential criterion which is equivalent to continuity**
Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Then f is continuous at c if and only if for all sequences (x_n) converging to c, the sequence $(f(x_n))$ converges and its limit is $f(c)$.

3. **The definition of cluster point**
If A is a subset of $\mathbb{R}$, then $c \in \mathbb{R}$ is a cluster point of A if for all $\delta > 0$ there is an element x of A not equal to c for which $|x - c| < \delta$.

4. **The definition of Cauchy sequence**
A sequence (x_n) of real numbers is a Cauchy sequence if for all $\varepsilon > 0$ there is an element $K(\varepsilon) \in \mathbb{N}$ so that for n and m in $\mathbb{N}$ with $n \geq K(\varepsilon)$ and $m \geq K(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

5. **The definition of convergence of a series and its sum**
Suppose $\sum_{j=1}^{\infty} a_j$ is an infinite series. Define the sequence of partial sums to be $x_n = \sum_{j=1}^{n} a_j$. Then the infinite series converges and its sum is L if the series (x_n) converges and its limit is L.

Part 2 of the Second Exam for Math 311, section 1

April 17, 2003

NAME ______________________________________

Do all problems, in any order.
No notes or texts may be used on this exam.
The last page contains the answers to Part 1.

Problem Number	Possible Points	Points Earned:
1	14	
2	12	
3	12	
4	12	
5	14	
6	16	
Total Points Earned:		