(14) 2. Suppose that f and g are bounded functions defined on the closed and bounded interval $[a, b]$ with $a < b$. Define $A = \sup\{f(x) + g(x) : x \in [a, b]\}$, $B = \sup\{f(x) : x \in [a, b]\}$, and $C = \sup\{g(x) : x \in [a, b]\}$. Prove that $A \leq B + C$. Give an example to show that A may be less than $B + C$.

(16) 3. a) If x is a real number, prove that x is the limit of some sequence (x_n) where each x_n is a rational number.

b) If $|x - 3| < 2$ find and verify some upper bound on $2x^3 + \dfrac{7x}{3x^2 + 4}$.

(12) 4. Let (x_n) be an infinite sequence. Assume $\lim(x_n) = L$. Let M be an element of $\mathbb{N}$ and suppose that the set S is defined by $S = \{x_n : n \geq M\}$. Prove that $\inf S \leq L$.

(18) 5. Assume the following for all parts of this problem:

> The function f is continuous on the closed and bounded interval $[a, b]$ with $a < b$, and f is always non-negative: $f(x) \geq 0$ for all $x \in [a, b]$.

a) Prove that $\int_a^b f$ is always non-negative.

Hint Take a *very* simple partition, and use a major result about continuous functions which you should cite carefully.

b) Suppose there is $c \in [a, b]$ with $f(c) > 0$. Prove that there is an interval of positive length where $f(x) > \frac{1}{2} f(c)$ for all x in the interval.

c) Suppose there is $c \in [a, b]$ with $f(c) > 0$. Prove that $\int_a^b f > 0$.

Hint Use your answer to b) and take an almost *very* simple partition.

(16) 6. Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a continuous function, and that $|g(x)| < 10$ for all $x \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3 + g(x)$. Prove that f has at least one real root. Cite any results you use carefully. Be sure that the function(s) involved satisfy all the hypotheses.

(20) 7. Give examples. You need *not* verify that your examples have the properties requested.

a) $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere *except* at 0 and 1.

b) $f : \mathbb{R} \to \mathbb{R}$ which is continuous only at 0 and 1.

c) A non-empty bounded subset of $\mathbb{R}$ which contains its sup and does not contain its inf.

d) $f : \mathbb{R} \to \mathbb{R}$ which is bounded and which is Riemann integrable in $[a, b]$ for any $0 \leq a < b$ and which is *not* Riemann integrable in $[c, d]$ for any $c < d \leq 0$.

e) An unbounded sequence which has subsequences converging to -1 and 1.

(16) 8. Suppose that $f(x) = x^2$ and $c \in \mathbb{R}$. Prove using the definition of continuity that f is continuous at c.

(16) 9. Suppose that (a_n) is a sequence of real numbers and that for each $n \in \mathbb{N}$, the following inequality is true: $|a_n - 4| < \frac{1}{n}$. Find a specific positive integer N so that for all $n \geq N$, $\left|\frac{1}{a_n} - \frac{1}{4}\right| < \frac{1}{100}$. Be sure to verify your claimed inequalities.

(18) 10. Suppose that p is a positive real number and that a sequence is defined recursively by

$$\begin{cases} x_1 = \sqrt{p} \\ x_{n+1} = \sqrt{p + x_n} \quad \text{for } n \in \mathbb{N} \end{cases}.$$

a) Prove that (x_n) is an increasing sequence.

b) Prove that (x_n) is bounded above by $1 + \sqrt{p}$.

c) Conclude that (x_n) converges, and find its limit, with brief explanation of your work.

(14) 11. Suppose f is a Riemann integrable function on $[0, 1]$. If $j \in \mathbb{N}$, let $a_j = \int_{\frac{1}{j+1}}^{\frac{1}{j}} f$. Prove that the infinite series $\sum_{j=1}^{\infty} a_j$ converges, and that its sum is $\int_0^1 f$.

Hint A Riemann integrable function must be bounded.

Answers to Part 1 of all three exams

The definition of supremum Suppose S is a nonempty subset of $\mathbb{R}$. x is a supremum of S if x is an upper bound of S (for all $s \in S$, $s \leq x$) and if y is any upper bound of S, then $x \leq y$.

A criterion for an upper bound to be a supremum Suppose x is an upper bound of a nonempty subset of $\mathbb{R}$. x is a supremum of S if and only if for all $\varepsilon > 0$, there is $s \in S$ with $x - \varepsilon < s \leq x$.

The Completeness Axiom If S is a nonempty subset of R which is bounded above, then S has a least upper bound.

The Archimedean Property If $x \in \mathbb{R}$ then there is $n \in \mathbb{N}$ so that $n > x$.

The definition of convergence of a sequence A sequence (x_n) converges to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ there is $K(\varepsilon) \in \mathbb{N}$ such that for $n \in \mathbb{N}$ with $n \geq K(\varepsilon)$, $|x_n - x| < \varepsilon$.

The definition of continuity Suppose $f \colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Then f is continuous at c if, given $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

A sequential criterion which is equivalent to continuity Suppose $f \colon \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Then f is continuous at c if and only if for all sequences (x_n) converging to c, the sequence $(f(x_n))$ converges and its limit is $f(c)$.

The definition of cluster point If A is a subset of $\mathbb{R}$, then $c \in \mathbb{R}$ is a cluster point of A if for all $\delta > 0$ there is an element x of A not equal to c for which $|x - c| < \delta$.

The definition of Cauchy sequence A sequence (x_n) of real numbers is a Cauchy sequence if for all $\varepsilon > 0$ there is an element $K(\varepsilon) \in \mathbb{N}$ so that for n and m in $\mathbb{N}$ with $n \geq K(\varepsilon)$ and $m \geq K(\varepsilon)$, $|x_n - x_m| < \varepsilon$.

The definition of convergence of a series and its sum Suppose $\sum_{j=1}^{\infty} a_j$ is an infinite series. Define the sequence of partial sums to be $x_n = \sum_{j=1}^{n} a_j$. Then the infinite series converges and its sum is L if the series (x_n) converges and its limit is L.

The definition of $US(f, \mathcal{P})$, the upper sum for f associated with the partition $\mathcal{P}$ on $[a, b]$ If $\mathcal{P} = \{a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b\}$ is a partition of the closed and bounded interval $[a, b]$, and f is a bounded function defined on $[a, b]$, then $US(f, \mathcal{P}) = \sum_{j=1}^{n} \sup\{f(x) : x \in [x_{j-1}, x_j]\}(x_j - x_{j-1})$.

A criterion directly involving upper and lower sums which is equivalent to Riemann integrability Suppose f is a bounded function defined on the closed and bounded interval $[a, b]$. f is Riemann integrable on $[a, b]$ if and only if for all $\varepsilon > 0$ there is a partition $\mathcal{P}$ so that $US(f, \mathcal{P}) - LS(f, \mathcal{P}) < \varepsilon$.

A version of the Fundamental Theorem of Calculus Suppose f is Riemann integrable on $[a, b]$. Then f is Riemann integrable on $[a, x]$ for all $x \in [a, b]$. Define $F(x) = \int_a^x f$ for $x \in [a, b]$. If f is continuous at $c \in [a, b]$ then F is differentiable at c and $F'(c) = f(c)$.

Part 2 of the Final Exam for Math 311, section 1

May 13, 2003

NAME ____

Do all problems, in any order.
No notes or texts may be used on this exam.
The last page contains the answers to Part 1.

Problem Number	Possible Points	Points Earned:
1	40	
2	14	
3	16	
4	12	
5	18	
6	16	
7	20	
8	16	
9	16	
10	18	
11	14	
Total Points Earned:		