**Useful information for the second exam in Math 311:01, Spring 2003**

The time, date, and place will be:

**SEC 212, Thursday, April 17, at 1:10 PM**
(The usual time & place on Thursdays.)

The exam will primarily cover the material of chapters 3, 4, and 5 as done by the class since the last exam. No books or notes may be used on the exam. The exam will have two parts.

**Part 1** will be worth 20 points out of 100. I will ask you to

- Define "$f$ is continuous at $c \in \mathbb{R}$."
  References: page 120 of the text or the course diary entry for 4/3/2003.
- State a sequential criterion which is equivalent to "$f$ is continuous at $c \in \mathbb{R}$."
  References: page 121 of the text or the course diary at the beginning of April.
- Define "$c$ is a cluster point of $A$."
  References: page 97 of the text or the course diary entry for 2/7/2003.
- Define "$(x_n)$ is a Cauchy sequence."
  References: page 81 of the text or the course diary entry for 3/12/2003.
- Define "An infinite series $\sum_{j=1}^{\infty} a_j$ converges and its sum is $L$."
  References: page 89 of the text or the course diary entry for 3/26/2003.

As soon as you finish and hand in part 1, I will give you part 2.

**Part 2** will contain the answers to the questions of part 1, and will have other questions to answer, similar to those below. The questions below are taken from exams given by various instructors of Math 311. I hope students will send me *plain text* e-mail with solutions to the problems with their names. I will proofread these messages and put them on the web. There are certainly more problems below than will appear on part 2 of the exam.

1. BECKHORN Let $f\colon A \to \mathbb{R}$ be a uniformly continuous function. Let $(x_n)$ be a Cauchy sequence in $A$. Prove that $(f(x_n))$ is a Cauchy sequence.
Note: You cannot use the argument $\lim(f(x_n)) = f(\lim(x_n))$ because $\lim(x_n)$ may not be an element of $A$.

2. BENSON Let $f\colon [0,3] \to \mathbb{R}$ be a continuous function. Suppose $f(0) - f(1) = f(2) - f(3)$. Prove that there exists $x \in [0,1]$ such that $f(x) + f(x+2) = 2f(x+1)$.

3. CHANG a) Find a positive number $b$ so that if $|x-1| < b$ then $|x^2 - 1| < \frac{1}{100}$. Prove your assertion.

b) Give an example of an unbounded sequence that has a convergent subsequence.

4. CITTERBART In this problem suppose that $(s_n)$ and $(t_n)$ are both increasing sequences and that for all $n \in \mathbb{N}$, $s_n \le t_n$.

a) Prove that if $(t_n)$ converges, then $(s_n)$ converges.

b) Prove that there are examples of sequences satisfying the hypotheses of this problem so that $(s_n)$ converges and $(t_n)$ does not converge.

**OVER**

5. **COHEN** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that satisfies the condition $|f(x) - f(y)| \le \sqrt{|x-y|}$ for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Prove that $f$ is uniformly continuous.

6. **GREENBAUM** Suppose $f\colon \mathbb{R} \to \mathbb{R}$, and $\lim\limits_{x\to 0} f(x) = L$. Let $g(x) = f(2x)$. Prove that $\lim\limits_{x\to 0} g(x) = L$.

7. **GUTHRIE** Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is a continuous bounded function: that is, there is $M > 0$ so that $|g(x)| \le M$ for all $x \in \mathbb{R}$. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) := x^2 + g(x)$ must have an absolute minimum in $\mathbb{R}$: that is, there must exist an $x_0 \in \mathbb{R}$ with $f(x_0) \le f(x)$ for all $x \in \mathbb{R}$.

8. **HEDBERG** Suppose $A = \{x : x > 0\}$. Assume $a < b < c$. Let $f\colon \to \mathbb{R}$ be a continuous function and let $(x_n)$, $(z_n)$ be sequences in $A$ such that $\lim(x_n) = 0$, $\lim(f(x_n)) = a$, $\lim(z_n) = 0$, and $\lim(f(z_n)) = c$.
Prove that there exists a sequence $(y_n)$ in $A$ such that $\lim(y_n) = 0$ and $\lim(f(y_n)) = b$.

9. **LEDUC** Suppose the domain of $f$ is all of $\mathbb{R}$ and that $\lim\limits_{h\to 0} f(x+h) - f(x-h) = 0$ for all $x \in \mathbb{R}$. Does this imply that $f$ is continuous? Explain your answer.

10. **MILES** Suppose that $c_j > 0$ for all $j \in \mathbb{N}$, and that $\sum\limits_{j=1}^{\infty} c_j$ converges absolutely. If $f\colon \mathbb{R} \to \mathbb{R}$ is a bounded function (that is, there is $M > 0$ so that $|f(x)| \le M$ for all $x \in \mathbb{R}$), explain carefully why the series $\sum\limits_{j=1}^{\infty} f(j)c_j$ converges. Must the series $\sum\limits_{j=1}^{\infty} f(c_j)$ also converge?

**Comment** Mr. Cohen pointed out that the problem as originally stated (without “absolutely”) is *false*! An example would probably be instructive.

11. **NUCCI** Let $(x_n)$ be a sequence. Assume $\lim(x_n) = x$. Suppose that $c$ is a cluster point of $\{x_n : n \in \mathbb{N}\}$. Prove: $x = c$.

12. **OLEYNICK** Let $c$ be a cluster point of a set $A$. Let $L$ be a real number. Let $f\colon A \to \mathbb{R}$ be a function such that $\lim\limits_{x\to c} f(x) = L$ and $f(x) \ne L$ for each $x \in A$. Prove that $L$ is a cluster point of $\{f(x) : x \in A\}$.

13. **SUNG** Suppose that $F\colon \mathbb{R} \to \mathbb{R}$ is defined by $F(x) := \begin{cases} x, & \text{if } x \text{ is rational} \\ -x, & \text{if } x \text{ is irrational} \end{cases}$.

a) Is $F$ continuous at 0? Prove your assertion.

b) Is $F$ continuous at 1? Prove your assertion.

14. **TROPEANO** Let $A$ and $B$ be subsets of $\mathbb{R}$ such that $A \cap B = \emptyset$. Let $c \in \mathbb{R}$ is such that $c$ is a cluster point of $A$ and $c$ is a cluster point of $B$. Let $f\colon A \to \mathbb{R}$ and $g\colon B \to \mathbb{R}$ be functions. Let the function $h\colon A \cup B \to \mathbb{R}$ be defined as follows: $h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$. Let $L$ be a real number and assume $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = L$. Prove that $\lim\limits_{x\to c} h(x) = L$.

I reserved a room for a review session. The time, date, and place are:

**Hill 425, Tuesday, April 15, at 6:10 PM**

I can review textbook problems, workshop problems, and try to give intellectual outlines (!) of sections of the course. Please: I hope that students will do the problems given here so that I won't need to.