Please write solutions to two of these problems. Hand them in next Wednesday, February 12. These written solutions should be accompanied by explanations using complete English sentences. Students may work alone or in groups. All students working in a group should contribute to the writeup and should sign what is handed in.

1. Suppose $\mathbb{F}$ is a field satisfying the algebra axioms. If a, b, c, d are elements of $\mathbb{F}$ with $c \neq 0$ and $d \neq 0$, prove the following results:

$$\frac{a}{c} + \frac{b}{d} = \frac{a \cdot d + b \cdot c}{c \cdot d} \qquad \frac{a}{c} \cdot \frac{b}{d} = \frac{a \cdot b}{c \cdot d} \qquad \frac{a \cdot d}{c \cdot d} = \frac{a}{c}.$$

Comment Remember that $\frac{A}{B}$ is an abbreviation for the product of A with the multiplicative inverse of B. Also remember that the phrase "multiplicative inverse" specifies an element of $\mathbb{F}$ only when B is not 0.

2. Suppose now that $\mathbb{F}$ is an ordered field, and that a and b are positive elements of $\mathbb{F}$. Prove that for every integer $n \in \mathbb{N}$, $a < b$ if and only if $a^n < b^n$.

Comment Exposition might be easier if you first proved the following implication by induction: if A and B are in $\mathbb{F}$ and if $0 < A < B$ and if $n \in \mathbb{N}$, then $A^n < B^n$. You could then *use* this result to help verify the statement above.

3. Suppose that x and y are elements of an ordered field, $\mathbb{F}$. Prove that

$$\frac{|x + y|}{1 + |x + y|} \leq \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}.$$

4. Suppose a and b are elements of an ordered field, $\mathbb{F}$. Prove: if for all positive $\varepsilon > 0$, $|b - a| < \varepsilon$ then $a = b$.

5. Suppose f is a function defined by the formula

$$f(x) = \frac{x^3 + 3x - 9}{x^4 + 5x^2 + 2}.$$

a) Use methods of this course to find numbers $0 < u \leq v$ so that if $x \in [0, 1]$, then $u \leq |f(x)| \leq v$. Prove your assertions.

b) Use methods of this course to find numbers $0 < p \leq q$ so that if $x \in [10, 11]$, then $p \leq |f(x)| \leq q$. Prove your assertions.

Comment You are *not* asked to find "best possible" numbers (whatever that may mean!). You *are* asked to find some numbers and prove certain assertions.