Please write solutions to two of these problems. Hand them in next Wednesday, February 19. These written solutions should be accompanied by explanations using complete English sentences. Students may work alone or in groups. All students working in a group should contribute to the writeup and should sign what is handed in.

1. Let A, B be nonempty subsets of $\mathbb{R}$ such that $A \cup B$ has an upper bound. Assume properties (I) and (II):

(I) For every $a \in A$ there exists $b \in B$ such that $a < b$.

(II) For every $b \in B$ there exists $a \in A$ such that $b < a$.

Prove that $\sup A = \sup B$. Give one example of a pair of such sets A and B.

2. Let A be a nonempty set of positive real numbers with property (III):

(III) For every $a \in A$ there exists $b \in A$ such that $b < (2/3)a$.

Prove that $\inf A = 0$. Also, give one example of such a set, A.

3. Let A_1, A_2, A_3, ... be an infinite sequence of nonempty subsets of $\mathbb{R}$ such that $B = \cup_{n \geq 1} A_n$ has an upper bound. Show that each A_n is bounded above. If $u_n = \sup A_n$, prove that $\sup B = \sup(\{u_1, u_2, u_3, ...\})$.

Also, give one example of a sequence of such sets, $\{A_n\}_{n \geq 1}$, with all of the u_n's distinct numbers and with $\sup B$ different from any of the u_n's.

4. Let A and B be bounded nonempty sets such that $\sup A = \sup B$. Prove that for every $a \in A$ and for every $\varepsilon > 0$ there exists $b \in B$ such that $a - \varepsilon < b$.

Also, give an example of two bounded sets A and B with $A \cap B = \emptyset$ and $\sup A = \sup B$.

5. Let A be a set of positive real numbers. Assume that A is nonempty. Assume also that A has property (IV):

(IV) For every $a \in A$ there exists $b \in A$ such that $b > a + \frac{1}{a}$.

Prove that A is not a bounded set. Also, give one example of such a set, A.

6. Let A be a set of positive real numbers. Assume that A is nonempty. Assume also that A has property (V):

(V) For every $a \in A$ there exists $b \in A$ such that $b < \frac{a}{1+a}$.

Prove that $\inf A = 0$. Also, give one example of such a set, A.

7. Prove the Betweenness Axiom:

The "Betweenness Axiom"

Suppose A and B are both non-empty subsets of $\mathbb{R}$, and that if $a \in A$ and $b \in B$, then $a \leq b$. Then there is at least one $c \in \mathbb{R}$ with $a \leq c \leq b$ for all $a \in A$ and $b \in B$.

Prove also that if the Betweenness Axiom is assumed in an ordered field, then the Completeness Axiom must be true.

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* Problems 1-6 are modifications of problems written by Professor Scheffer. Problem 7 was written following a suggestion of Professor Saks.