Problems for 311:01

The last one entirely about sequences!

Please write solutions to two of these problems. Hand them in Monday, March 31. The written solutions should be accompanied by explanations using complete English sentences. Students may work together but should hand in independent writeups. *Students may ask me questions.*

1. Find all starting values x_1 such that the sequence defined by $x_{n+1} = x_n^2$ converges to a limit L and prove that this convergence takes place. Prove that you have found all such choices of x_1.

2. a) Either find an example (and briefly verify the desired properties of the example) or explain why one can't exist (citing theorems to prove your assertions).

   i) Find an example of a monotone sequence which is not Cauchy.

   ii) Find an example of a monotone sequence which is not bounded.

   iii) Find an example of a Cauchy sequence which is not monotone.

   iv) Find an example of a Cauchy sequence which is not bounded.

   v) Find an example of a bounded sequence which is not monotone.

   vi) Find an example of a bounded sequence which is not Cauchy.

b) Which subsets of the three attributes $\{$bounded, Cauchy, monotone$\}$ imply that a sequence *must* converge? (There are a total of $2^3 = 8$ such subsets. For each of them, supply an example or a citation to verify your assertion.)

3. A tired frog hops on the real line. The frog hops to the left or to the right, and the direction varies unpredictably. Each hop tires the frog more, and therefore each hop is *at most* half the length of the previous hop. Besides this restriction, the actual length of the hop varies unpredictably. Does the sequence of positions of the frog converge to a point in the line? (Explain your conclusion in detail, substantiating your reasoning in the language of this course.)

4. (In the following problem C is some unspecified positive number.)

a) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{n \cdot m}.$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

b) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{n + m}.$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

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c) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{n} + \frac{C}{m}.$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?
d) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{\max(n, m)}.$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?

e) Suppose (a_n) is a sequence with the following property:

$$\text{For every } n \text{ and } m \text{ in } \mathbb{N},\ |a_n - a_m| < \frac{C}{\min(n, m)}.$$

Prove that (a_n) is a Cauchy sequence and that it must therefore converge. Find an example of such a sequence. What can you say about its limit?