Please write solutions to two of these problems. Hand them in Thursday, April 10. The written solutions should be accompanied by explanations using complete English sentences. Students may work together but should hand in independent writeups. *Students may ask me questions.*

1. a) Suppose f and g are continuous on all of $\mathbb{R}$, and that $f(x) = g(x)$ for all rational numbers. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

b) Show that if one of the functions in a) is *not* continuous, then f and g may not be equal for all $x \in \mathbb{R}$.

2. a) Suppose that h's domain is $\mathbb{R}$ and that for all $x \in \mathbb{R}$, $|h(x) - 17| \leq 56\sqrt{|x-5|}$. Prove that $\lim_{x \to 5} h$ exists and that the value of the limit is 17.

b) Suppose that j's domain is $\mathbb{R}$ and that there are L, a, K, and $p \in \mathbb{R}$ with K and p positive so that for all $x \in \mathbb{R}$, $|j(x) - L| \leq K|x-a|^p$. Prove that $\lim_{x \to a} j$ exists and that the value of the limit is L.

c) Is part a) or part b) easier?

3. In this problem suppose that F has domain all of $\mathbb{R}$ and that $\lim_{x \to 0} F = L$.

a) If $G(x) = F(4x)$, prove that $\lim_{x \to 0} G(x) = L$.

b) If a is a non-zero constant, and if $H(x) = F(ax)$, prove that $\lim_{x \to 0} H(x) = L$.

c) If $K(x) = F(x^2)$, prove that $\lim_{x \to 0} K(x) = L$.

4. Suppose that f and g both have domains all of $\mathbb{R}$, that $\lim_{x \to a} f$ and $\lim_{x \to a} g$ exist, and that both of the limits are L. Define a function h whose domain is all real numbers except a by $h(x) = \begin{cases} f(x), & \text{if } x < a \\ g(x), & \text{if } x > a \end{cases}$. Prove that $\lim_{x \to a} h$ exists and is equal to L.

5. Let c be a cluster point of a set A. Prove that either I) or II) must be true:

I) There exist $a_1 < a_2 < a_3 < \cdots$ such that $\lim(a_n) = c$ and $a_n \in A$ for all $n \in \mathbb{N}$.

II) There exist $a_1 > a_2 > a_3 > \cdots$ such that $\lim(a_n) = c$ and $a_n \in A$ for all $n \in \mathbb{N}$.

6. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function such that $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove: there exist $A \subseteq \mathbb{R}$ and $g\colon A \to \mathbb{R}$ such that 0 is a cluster point of A,. $g(x) = f(x)$ if $x \in A$, and $\lim_{x \to 0} g$ exists.

Hint Use a theorem of Great^6 Grandfather to prove that the sequence $f(\frac{1}{1}), f(\frac{1}{2}), f(\frac{1}{3}), \ldots$ must have a convergent subsequence $f(\frac{1}{n_1}), f(\frac{1}{n_2}), f(\frac{1}{n_3}), \ldots$ and this leads to A and g.

Note: Problems 5 and 6 are versions of problems written by Professor Scheffer.