Shakespeare's birthday!

Problems for 311:01

4/23/2003

Please read all of these problems and write solutions to two of them. Hand in your work on Wednesday, April 30.

1. Suppose f is a Riemann integrable function on $[a,b]$, and g is a function disagreeing with f at one point. That is, there is $c \in [a,b]$ with $f(c) \neq g(c)$, but $f(x) = g(x)$ for all $x \in [a,b] \setminus \{c\}$. Prove that g is Riemann integrable on $[a,b]$, and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$.

2. Suppose that f is a Riemann integrable function on $[a,b]$. Define the function g on $[a,b]$ by $g(x) = |f(x)|$, so g is the absolute value of f. Prove that g is also Riemann integrable on $[a,b]$, and that $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b g(x)\,dx$.

Comment The inequality is a sort of integrated form of the Triangle Inequality.

3. a) Suppose that f is a Riemann integrable function on $[a,b]$, and that $f(x) \geq 0$ for all $x \in [a,b]$. Give an example to show that $\int_a^b f(x)\,dx$ can be 0 even if the function values are not always 0.

b) Suppose that f is a continuous function on $[a,b]$, and that $f(x) \geq 0$ for all $x \in [a,b]$. If $\int_a^b f(x)\,dx = 0$, show that $f(x) = 0$ for all $x \in [a,b]$.

4. Create a sequence of Riemann integrable functions, $\big(f_n(x)\big)$, defined on the interval $[0,1]$, having the following properties:

i) For all $n \in \mathbb{N}$ and all $x \in [0,1]$, $f_n(x) \geq 0$.

ii) For all $x \in [0,1]$, the sequence $\big(f_n(x)\big)$ converges to 0.

iii) For all $n \in \mathbb{N}$, $\int_0^1 f_n(x)\,dx = 1$.

Thus $\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$ and $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} 1 = 1$.

Comment Simple examples are better than complicated examples. Try step functions (just one step each!). The object of this problem is to show you that limit and integral can't be "exchanged" all the time.

5. a) Suppose f is a continuous function on $[a,b]$. Show that there must exist at least one $c \in [a,b]$ so that $\int_a^b f(x)\,dx = f(c)(b-a)$.

b) Give an example to show that if f is a Riemann integrable function on $[a,b]$, there may not be *any* $c \in [a,b]$ so that $\int_a^b f(x)\,dx = f(c)(b-a)$.

Comment The result in a) is sometimes called the *Mean Value Property for Integrals*. It is true for continuous functions, but not always true for integrable functions.