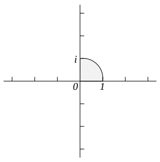
(16) 1. W is the set of complex numbers z satisfying the inequalities $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z \geq 0$ and $|z| \leq 1$. So W is the complex numbers in the first quadrant inside the closed unit circle. This problem investigates the effect of the mapping $z \to \overline{3iz^2 + 2}$ on W. Complete every part of this problem in order to get full credit.

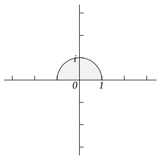
a) Sketch W on the axes indicated.



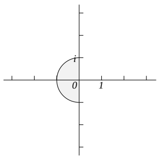
b) Is 0 an interior point of W?

Answer No, because no disc centered at 0 with positive radius is entirely inside W.

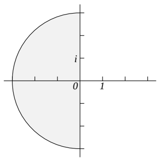
c) Suppose W_1 is the image of W under the mapping $z \to z^2$. Sketch W_1 on the axes indicated.



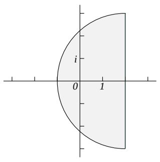
d) Suppose W_2 is the image of W_1 under the mapping $z \to iz$. Sketch W_2 on the axes indicated.



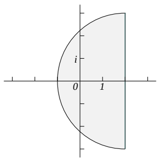
e) Suppose W_3 is the image of W_2 under the mapping $z \to 3z$. Sketch W_3 on the axes indicated.



f) Suppose W_4 is the image of W_3 under the mapping $z \to z+2$. Sketch W_4 on the axes indicated.



g) Suppose W_5 is the image of W_4 under the mapping $z \to \overline{z}$. Sketch W_5 on the axes indicated.



h) W_5 is the image of W under the mapping $z \to \overline{3iz^2 + 2}$. Is 0 an interior point of W_5?

Answer Yes. For example, a disc of radius 1 centered at 0 is contained entirely within W_5.

(16) 2. a) Find all values of $(-i)^i$.

Answer $(-i)^i = \exp\left(i \log(-i)\right) = \exp((i\left(\ln(|-i|) + i\arg(-i)\right)) = \exp(\left(i\left(\ln(1) + i(-\frac{\pi}{2} + 2k\pi\right)\right)) = e^{\frac{\pi}{2} - 2k\pi}$ where k is any integer.

b) Suppose that b is real and a is any non-zero complex number. Prove that $\left|a^b\right| = \left||a|^b\right|$ for all values of both sides of the equation.

Answer $a^b = \exp(b(\ln|a| + i\arg(a))) = e^{b\ln(|a|)} \cdot e^{ib\arg(a)}$. Since b is real, the exponent in the first factor is real and the exponent in the second factor is imaginary. The second factor then has modulus $= 1$, so that $|a^b| = e^{b\ln(|a|)}$. Notice that if we change a to $|a|$ the answer becomes $e^{b\ln(|(|a|)|)}$ which equals $e^{b\ln(|a|)}$ so that we have proved what was asked.

(16) 3. a) Suppose $c(x, y)$ is a harmonic function, so $\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0$. Prove that $f = \frac{\partial c}{\partial x} - i\frac{\partial c}{\partial y}$ is complex analytic.

Answer We check THE CAUCHY-RIEMANN EQUATIONS. Here $u = c_x$ and $v = -c_y$. The first equation is $u_x = v_y$ which is true if $(c_x)_x = (-c_y)_y$; that happens since $\triangle c = 0$. The second equation is $u_y = -v_x$ which is true if $(c_x)_y = -(-c_y)_x$; this occurs because $c_{xy} = c_{yx}$: mixed second order partial derivatives are equal.

b) Verify that $w(x, y) = \cos x \cosh y$ is harmonic, and find one harmonic conjugate.

Answer $w_{xx} = -\cos x \cosh y$ and $w_{yy} = \cos x \cosh y$, so $\triangle w = 0$. If we want q so $w + iq$ is complex analytic, we need to find q so that $\begin{cases} q_x = -w_y & = -\cos x \sinh y \\ q_y = \ \ w_x & = -\sin x \cosh y \end{cases}$ (THE CAUCHY-RIEMANN EQUATIONS again). We can guess: $q(x, y) = -\sin x \sinh y$ will do. Actually, the complex analytic function $\cos x \cosh y - i \sin x \sinh y$ is $\cos(x + iy) = \cos(z)$.

(16) 4. a) Compute $\int_C 2x + 3iy^2\, dz$ where C is the straight line segment from 0 to $1 + i$.

Answer PARAMETERIZE If $z = x + iy = t + it$ with $0 \leq t \leq 1$ then $dz = (1 + i)dt$. The integral becomes $\int_0^1 (2t + 3it^2)(1 + i)\, dt = \int_0^1 (2t - 3t^2) + i(2t + 3t^2)\, dt = 2i$.

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b) If C is the circle of radius 2 centered at 0, verify that $\left|\int_C \left(\frac{z+3}{z^3+1}\right)dz\right| \le \frac{20\pi}{7}$. Show details of your estimates.

Answer We overestimate the modulus of the integral by ML where L, the length, is (2π)radius $= (2\pi)2 = 4\pi$, and M is an overestimate of the integrand on the circle. The integrand is a quotient. We estimate it by first overestimating the top: $|z+3| \overset{\triangle}{\le} |z|+3 = 2+3 = 5$ on the circle. We next underestimate the bottom: $|z^3+1| \overset{\nabla}{\ge} |z^3| - 1 = |z|^3 - 1 = 2^3 - 1 = 7$ on the circle. Therefore the modulus of the integrand is estimated by $\frac{5}{7}$ and $ML = \frac{5}{7}\cdot 4\pi = \frac{20\pi}{7}$ as desired.

(16) 5. In all parts of this problem, C is the unit circle oriented in the standard counterclockwise manner. Evaluate the integrals with reasons for the answers.

a) $\int_C \frac{1}{z}\,dz$. **Answer**

PARAMETERIZE $z = e^{it}$, $dz = ie^{it}\,dt$ with $0 \le t \le 2\pi$ gives $\int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\,dt = \int_0^{2\pi} i\,dt = 2\pi i$.

THE CAUCHY INTEGRAL FORMULA applied to the complex analytic function "1" evaluated at 0 shows that the integral must be 0.

b) $\int_C \frac{1}{z^2}\,dz$. **Answer**

PARAMETERIZE If $z = e^{it}$ and $0 \le t \le 2\pi$, then $dz = ie^{it}\,dt$ and the integral becomes $\int_0^{2\pi} \frac{1}{e^{2it}} ie^{it}\,dt = \int_0^{2\pi} ie^{-it}\,dt = \frac{i}{-i}e^{-it}\Big]_0^{2\pi} = (-1)1-(-1)1 = 0$.

ANTIDERIVATIVE Any function with an antiderivative in an open set containing a closed curve must have integral 0 around the curve. Since $\frac{d}{dz}\left(\frac{-1}{z}\right) = \frac{1}{z^2}$ if $z \ne 0$ and the unit circle doesn't go through 0, the integral must be 0.

THE CAUCHY INTEGRAL FORMULA FOR THE FIRST DERIVATIVE of the complex analytic function "1" evaluated at 0 shows that the integral must be 0.

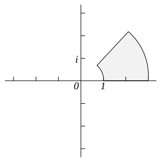
c) $\int_C \frac{e^{5z}}{z}\,dz$.

Answer $(2\pi i)e^{5\cdot 0} = 2\pi i$ using THE CAUCHY INTEGRAL FORMULA.

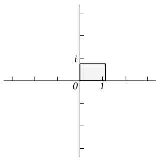
d) $\int_C \frac{e^{5z}}{z^2}\,dz$.

Answer $(2\pi i)\frac{d}{dz}\left(e^{5z}\right)\big|_{z=0} = (2\pi i)5e^{5\cdot 0} = 10\pi i$ using THE CAUCHY INTEGRAL FORMULA FOR THE FIRST DERIVATIVE.

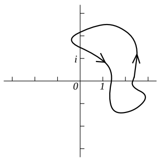
(8) 6. a) Sketch the region of the complex numbers satisfying the inequalities $1 \le |z| \le 3$ and $0 \le \arg z \le \dfrac{\pi}{4}$.



b) Sketch the image of the region in part a) under the mapping $z \to \operatorname{Log} z$, the *principal branch* of logarithm.

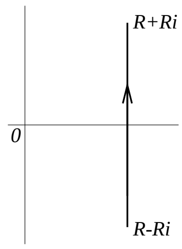


c) Compute $\int_C \operatorname{Log} z\,dz$ where C is the curve shown. Briefly explain your answer.



Answer $\int_C \operatorname{Log} z\,dz = 0$ using CAUCHY'S THEOREM, since C is a simple closed curve and Log is complex analytic inside and on C.

(12) 7. Suppose C_R is the vertical line segment from $R - Ri$ to $R + Ri$ as shown, and $f(z) = \dfrac{e^{(-z^2)}}{z^4+1}$. Prove that $\displaystyle\lim_{R\to+\infty} \int_{C_R} f(z)\,dz = 0$. Show details of your estimates.



Answer We overestimate the modulus of the integral by ML, where L is the length of C_R and M is an overestimate of the modulus of f on C_R. Surely $L = 2R$. Since f is a quotient, we again estimate its modulus by overestimating the modulus of the top and underestimating the modulus of the bottom.

The top is $\exp(-z^2)$. If z is on C_R, then z^2 has real part ≥ 0 so that $-z^2$ has real part ≤ 0. But $|\exp(w)| = \exp(\operatorname{Re} w)$, so if z is on C_R, $\exp(-z^2) \le \exp(0) = 1$. The top is overestimated by 1.

The bottom is z^4+1. Routinely, $|z^4+1| \overset{\nabla}{\ge} |z^4| - 1 = |z|^4 - 1 \ge R^4 - 1$ since points on C_R have distance at least R from 0.

Thus for z on C_R, $|f(z)| \le \frac{1}{R^4-1}$. The modulus of the integral is $\le \frac{2R}{R^4-1}$ which $\to 0$ as $R \to \infty$.

Note This problem can also be done by parameterizing the contour integral. Checking that the resulting Riemann integral has limit $= 0$ requires similar algebraic manipulations and estimations.