Review Problems for the first exam in Math 403; spring 1999

1. **Elementary computations** Evaluate the following expressions.

a) $\operatorname{Re}\left(\frac{1+i}{1-i}\right)$ b) $\operatorname{Im}\left(\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)^9\right)$ c) $\log(-1+i)$ d) $\operatorname{Log}(-e)$

2. **More computations** Solve for all possible z, expressing your answers in rectangular form.

a) $e^z = 1$ b) $\cos z = 0$ c) $\operatorname{Log} z = 1$ d) $z^4 = -16$ e) $\sin z = 2$

3. **Differentiability** a) Show that $g(z) = z\overline{z}$ is differentiable at 0 and is *not* differentiable at 1. Where is g complex analytic?

b) Suppose that u and v are the real and imaginary parts of a complex analytic function, h. Suppose that the range of h always lies on the hyperbola $uv = 1$. Prove that h must be constant.

4. **Mapping** Consider the set of complex numbers which lie on and inside the triangle T whose vertices are 0 and 1 and $1+i$. Sketch with justification the image of this set under the following complex analytic functions:

a) z^2 b) $iz+2$ c) $\exp z$ d) $\operatorname{Log} z$* e) $\sqrt{z}$

5. **Harmonicity** a) Select constants A and B so that $x^3+2x^2y+Axy^2+By^3$ is harmonic, and find *all* harmonic conjugates of this function.

b) Verify that $\arctan\left(\frac{2xy}{x^2-y^2}\right)$ is harmonic**

6. **Elementary functions** a) Use the exponential definition of sin and cos to verify that $\sin 2z = 2\sin z\cos z$ for all complex z.

b) Find an example of z with $\operatorname{Arg}(z^2) \neq 2\operatorname{Arg} z$. Give some domain of z's where the two quantities *are* equal, and explain why.

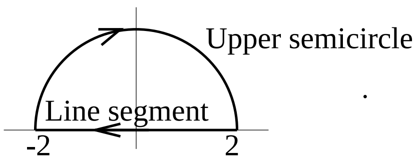
c) Suppose $H(z) = \operatorname{Log}\left(z^2-1\right)$. Compute $H(2)$ and $H(\frac{1+i}{\sqrt{2}})$. Describe the largest domain containing both 2 and $\frac{1+i}{\sqrt{2}}$ in which H is complex analytic. Write explicit formulas for the real and imaginary parts of H in terms of elementary functions.

d) Describe the behavior of $\sin z$ as $z \to \infty$ along the positive imaginary axis.

e) Define the principal determination of $(z-1)^{1/2}$. Where is this function analytic?

7. **Complex contour integrals** Compute the following integrals.

a) $\int_C (2x-iy)\,dz$ where C is the curve $y = x^2$ for $0 \leq x \leq 2$.

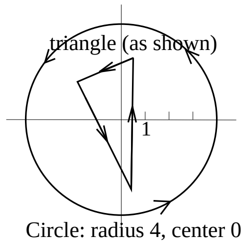
b) $\int_C |z|^2\,dz$ where C is 

.

* Is there a problem? Hmmm ...

** Please don't do a direct computation!

c) $\int_C \frac{\cos z}{z^2-\pi^2}\, dz$ where C is each one of the contours shown:
(One way: use partial fractions on the irritating part!)

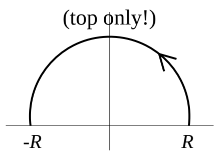


d) $\int_{|z|=1} \frac{e^z}{z^2-4}\, dz$.

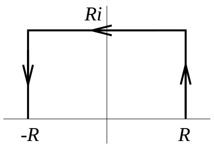
8. **Integral Estimation** a) Suppose C_R is the circle $|z| = R$ ($R > 1$), described in the counterclockwise fashion. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \operatorname{Log} R}{R} \right) .$$

b) Show that $\lim_{R\to\infty} \int_{C_R} \frac{z^2-1}{z^4+5}\, dz = 0$ where C_R is



c) Show that $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}}{z^2-3}\, dz = 0$ where C_R is



9. **Tricks?*** a) Suppose C is a simple closed contour. Compute the integrals $\int_C \frac{z^{10}+e^{iz}}{z-w}\, dz$ and $\int_C \frac{z^{10}+e^{iz}}{(z-w)^4}\, dz$ when w is *not* on C. There will be different answers, depending on whether w is inside or outside of C.

b) Suppose C is the unit circle and a is a real number. Explain why $\int_C \frac{e^{az}}{z}\, dz = 2\pi i$. Then express the integral parametrically to verify that $\int_0^\pi e^{a\cos t} \cos(a \sin t)\, dt = \pi$**.

* Well, profound tricks!

** Try this on Maple!