Review Problems for the final exam in Math 403; Spring, 1999

Please Look again at the review problems for the first exam and look again at your first exam. If we thought it was important before, we may well ask about it again on the final! The problems below were almost all *selected* from previous finals in this course to reflect what was done in the last half of the course. The final exam will cover the whole course, although there will be more emphasis on recently discussed material.

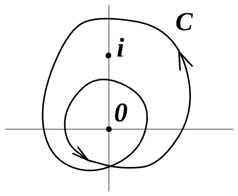
1. **Some real integrals** Compute the following integrals using the method of residues. In each case show clearly the contour of integration and explain why certain integrals tend to 0 in the limit.

a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$ b) $\displaystyle\int_{0}^{\infty} \frac{x\sin ax}{x^2+9}\,dx,\ a\neq 0$ c) $\displaystyle\int_{0}^{2\pi} \frac{dx}{2+\sin x}$

2. **Laurent expansions** a) The function $f(z) = \dfrac{1}{z^2-4z}$ has two Laurent series expansions about the origin, valid in different regions. What are these regions, and what are the expansions?

b) What is the Laurent series expansion in powers of $z-2$ of the function $\dfrac{1}{(z-2)(z-5)}$ at $z=4$? What is the largest region in which this series converges, and what is the sum of this series?

2. **Residue Integrals** a) Suppose $f(z)$ is entire with $f(0)=1$, $f'(0)=2$, $f''(0)=4$, $f(i)=3$, $f'(i)=-1$, and $f''(i)=-5$. If C be the curve shown, find $\displaystyle\int_C \frac{f(z)}{z}\,dz$ and $\displaystyle\int_C \frac{f(z)}{(z-i)^2}\,dz$.



b) Evaluate $\displaystyle\int_C \frac{dz}{e^z-1}$ where C is the circle of radius 9 and center 0.

3. **Isolated Singularities** For each of the following functions, identify whether the given singularity $z=z_0$ is removable, essential, or a pole. If a pole, give the order. In every case, determine the residue at z_0, $\operatorname*{Res}_{z=z_0}$. Show your reasoning.

a) $\dfrac{e^z(1-3\sin z)}{z^2}$, $z_0=0$. b) $\sin\left(\dfrac{1}{2z}\right)$, $z_0=0$. c) $\dfrac{(\sin z)^2}{\mathrm{Log}z(1-\cos z)}$, $z_0=2\pi$.

4. **Estimation** a) State Cauchy's <u>estimates</u> for the n^{th} derivative of a complex analytic function. Show then that there cannot be an analytic function $f(z)$ in $|z|\leq 1$ which satisfies $f^{(n)}(0)=n\cdot n!$ for *all* n.

b) Give an example of an entire function with $f(0) = 2$ and $|f(z)| \geq 1$ for all z. Is your example the only possible one?

5. **True or false** If true, give a brief explanation. If false, correct what is written and give an example to show why the original statement is false. *Read the statements carefully!*

a) If $f(z)$ is entire, the mapping $w = f(z)$ preserves angles.

b) If $f(z)$ is complex analytic in a domain D and C is a simple closed contour in D, then $\int_C f(z)\,dz = 0$.

c) Suppose $f(z)$ is complex analytic in a domain D and z_0 is in D. Then the Taylor series of $f(z)$ centered at z_0 converges to $f(z)$ for every z in D.

6. **Sneaky stuff** a) Suppose that f and g are entire functions and $f(z) = g(z)$ at each point of the circle $|z| = 2$. Explain why it follows that $f(1+i) = g(1+i)$.

b) What is $\displaystyle\max_{\frac{1}{2} \leq |z-1| \leq \frac{3}{2}} \left| \frac{z^2 - 4z + 4}{z - 1} \right|$?

c) If $f(z) = z^{10} \sin(z^2)$, what is $f^{(20)}(0)$?

d) Suppose T is the inside of the triangle whose vertices are 0, 1, and i, and S is the inside of the rectangle whose vertices are $\pm 2 \pm 2i$. If $f(z)$ is complex analytic on S then the maximum of $|f'(z)|$ for z in T is less than or equal to the maximum of $|f(z)|$ for z in S.

Hint Begin by writing some complex variables formula connecting f' and f.

7. **Linear fractional transformations** a) Find the linear fractional transformation mapping $-i$ to -1, 0 to i, and i to 1.

b) Into what curve is the imaginary axis $x = 0$ mapped by this transformation?

c) Show that any linear fractional transformation which is not the identity transformation $w = z$ has at most two fixed points in the extended plane.

8. **Mapping** For two of the following three mappings $w = f(z)$, describe how the half-strip $0 < x < \frac{\pi}{2}$, $y > 0$ in the z-plane gets mapped into the w-plane. Discuss in particular the images of the vertical half-lines $x =$constant, $y > 0$, and the angles they make with the image of the base $0 < x < \frac{\pi}{2}$, $y = 0$.

$$\text{a) } w = e^z \qquad \text{b) } w = \frac{1}{z} \qquad \text{c) } w = \sin z$$

9. **Rouché** If n is any integer greater than 2, then all the roots of $z^n - \dfrac{(z^2 + z + 1)}{4}$ lie inside the unit circle.