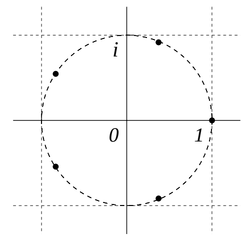
640:403:05 **Answers to the First Exam** **3/1/2001**

(6) 1. Find the principal argument $\operatorname{Arg} z$ when $z = (2-2i)^7$.

Answer $2-2i$ is in the fourth quadrant, and one value if its argument is $-\frac{\pi}{4}$. One value of the argument of its seventh power is $-\frac{7\pi}{4}$.The "Arg" of that number must be in the interval between $-\pi$ and π (including π), so $\operatorname{Arg} z$ is $\frac{\pi}{4}$.
Alternatively, $z = 2e^{-i\pi/4}$ so $z^7 = 2^7 e^{-7i\pi/4}$, leading to the same answer.

(12) 2. Describe all solutions of $z^5 = 1$ algebraically in either rectangular or polar form. Sketch these solutions on the axes provided.

Answer If $z^5 = 1$ and $z = re^{i\theta}$, then $z^5 = r^5 e^{5i\theta}$. If $r^5 = 1$, with r a non-negative real, then $r = 1$. 5θ must be 0 (mod 2π), which leads to the list $z = e^{2\pi ik/5} = \cos(2\pi k/5) + i\sin(2\pi k/5)$ for integers k in the set $\{0,1,2,3,4\}$. There are "simple" rectangular expressions for these complex numbers involving square roots of small integers. I don't know them.* The picture I want will indicate the vertices of a regular pentagon inscribed in the unit circle with one vertex at 1. The central angle between the vertices is $2\pi/5$ or 72°.



(12) 3. Suppose f is an analytic function with real part $u(x,y)$ and imaginary part $v(x,y)$. Explain why the function $H(x,y) = \big(u(x,y)\big)^2 - \big(v(x,y)\big)^2$ is harmonic, and find a harmonic conjugate of $H(x,y)$ in terms of $u(x,y)$ and $v(x,y)$.

Answer $Q(z) = z^2 = x^2 - y^2 + i(2xy)$ is analytic, and since composition of analytic functions is analytic, $Q(f(z))$ is analytic. The real part of $Q(f(z))$ is $H(x,y)$, and since the real part of an analytic function is harmonic, $H(x,y)$ is harmonic. The imaginary part of $Q(f(z))$ will be a harmonic conjugate of $H(x,y)$, and this is $2u(x,y)v(x,y)$.

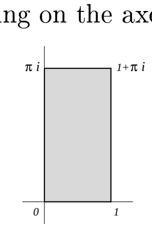
An answer can also be obtained through a direct computation of the Laplacian of H, use of the Cauchy-Riemann equations, etc., but the computation is elaborate, allowing many opportunities for error. Several very persistent students solved the problem this way correctly. A solution might go like this:

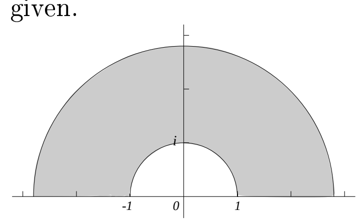
First derivatives	Second derivatives	Use of the Cauchy-Riemann equations
$\begin{cases} H_x = 2uu_x - 2vv_x \\ H_y = 2uu_y - 2vv_y \end{cases}$	$\begin{cases} H_{xx} = 2u_xu_x + 2uu_{xx} - 2v_xv_x - 2vv_{xx} \\ H_{yy} = 2u_yu_y + 2uu_{yy} - 2v_yv_y - 2vv_{yy} \end{cases}$	$\begin{cases} H_{xx} = 2u_xu_x + 2uu_{xx} - 2u_yu_y - 2vv_{xx} \\ H_{yy} = 2u_yu_y + 2uu_{yy} - 2u_xu_x - 2vv_{yy} \end{cases}$

Then add, and notice that since u and v are both harmonic (so $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$) that $H_{xx} + H_{yy} = 0$. Some further careful effort is needed to find a harmonic conjugate of H. If K is such a function, then $K_x = -H_y = -(2uu_y - 2vv_y)$ and $K_y = H_x = 2uu_x - 2vv_x$ so again using the Cauchy-Riemann equations, $K_x = -(-2uv_x - 2vu_x) = 2uv_x + 2vu_x$ and $K_y = 2uv_y - 2v(-u_y) = 2uv_y + 2vu_y$. An inspired guess (?) shows that $K = 2uv$ satisfies these equations.

(16) 4. a) Suppose S is the rectangular region whose boundary is the rectangle with corners 0, 1, πi, and $1+\pi i$ as shown. Sketch the image of S under the exponential mapping on the axes given.

Answer The modulus of e^z is $e^{\operatorname{Re} z}$, so that the image of S under the exponential map varies from e^0 to $e^1 \approx 2.7$. The argument of e^z is $\operatorname{Im} z$, so the argument varies from 0 to π. The result is the half-annulus shown.





b) Find <u>all</u> solutions of $(e^z)^2 = -1$.

Answer Look for z so that $e^z = i$ or $e^z = -i$. If $e^z = i$, then $z = \log i = \ln|i| + i\arg i$ so z must be $i\left(\frac{\pi}{2} + 2\pi k\right)$ for any integer k. One solution (with $k=0$) can be seen in the answer for the previous part. Similarly, the solutions of $e^z = -i$ are $i\left(\frac{-\pi}{2} + 2\pi k\right)$ for any integer k.

OVER

* Since $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$, one root is 1. `Maple` reports that the other roots are $\frac{1}{4}\sqrt{5} - \frac{1}{4} + \frac{1}{4}i\sqrt{2}\sqrt{5+\sqrt{5}}$, $-\frac{1}{4}\sqrt{5} - \frac{1}{4} + \frac{1}{4}i\sqrt{2}\sqrt{5-\sqrt{5}}$, $-\frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4}i\sqrt{2}\sqrt{5-\sqrt{5}}$, and $\frac{1}{4}\sqrt{5} - \frac{1}{4} - \frac{1}{4}i\sqrt{2}\sqrt{5+\sqrt{5}}$.

(10) 5. Find the two points in the complex plane where the function $\left(\frac{1}{4}x^4 + y^2\right) + i(x^2y)$ is complex differentiable. Where is this function analytic?

Answer The function is complex differentiable where the Cauchy-Riemann equations are satisfied. Here $u(x,y) = \left(\frac{1}{4}x^4 + y^2\right)$ and $v(x,y) = x^2y$. $u_x = v_y$ becomes $x^3 = x^2$ and solutions of this are $x = 0$ and $x = 1$. $u_y = -v_x$ is $2y = -2xy$ here. When $x = 0$ we see that $y = 0$. When $x = 1$, $y = 0$. The two points are 0 and 1. The function is not complex differentiable in any open set so it is analytic nowhere.

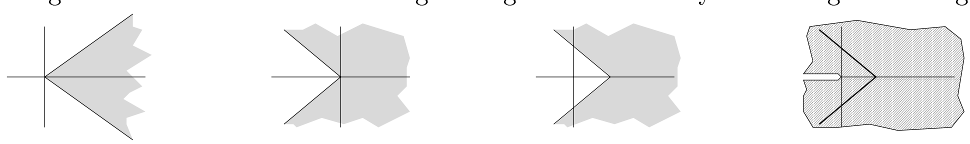
(6) 6. Describe with some justification what happens to $\sin z$ as $z \to \infty$ along the positive imaginary axis.

Answer $\sin z = \sin(x + iy) = \sin x \cos(iy) + \cos x \sin(iy)$. When $x = 0$ the equation becomes $\sin(iy) = i \sinh(y)$. As $y \to +\infty$, $\sinh y$ increases steadily without any upper bound. So on the positive imaginary axis as $y \to +\infty$, $\sin z$ is imaginary, increasing, and unbounded.

(12) 7. Describe a connected open subset W of the plane that includes the positive real axis on which an analytic branch of $\log(z^3 + 5)$ can be defined. Justify your assertion with an explanation. **Hint** A simple wedge might be a good candidate.

Answer Log is analytic in the complex plane with the non-positive real axis deleted. If we can find a domain for $z^3 + 5$ which maps into that part of the plane and includes the positive real axis we will be done. For example, consider the "wedge" W defined by requiring that $\operatorname{Arg} z$ be greater than $-\frac{\pi}{4}$ and less than $\frac{\pi}{4}$. This is a connected open set containing the positive real numbers. Then cubing maps that wedge into the set with $\operatorname{Arg} z$ between $-\frac{3\pi}{4}$ and less than $\frac{3\pi}{4}$. Adding 5 to points in this set moves them to the right 5 units – still in the set. And the result is inside the domain where Log is analytic.

Starting with W The effect of cubing   Right translation by 5   Sitting inside Log's domain



(16) 8. a) Compute $\int_C (x^2 + iy)\, dz$ where C is the straight line segment from 0 to $1 + 2i$.

Answer Parameterize by $z = x + iy = (1+2i)t$ with $0 \le t \le 1$. Then $dz = (1+2i)\, dt$ and $x = t$ and $y = 2t$, so that $x^2 + iy = t^2 + 2it$. The contour integral becomes $\int_0^1 (t^2 + 2it)(1+2i)\, dt = \left(\frac{1}{3} + i\right)(1+2i)$. This answer is fine. If you must "simplify", you should get $-\frac{5}{3} + \frac{5}{3}i$.

b) If C is the circle of radius 2 centered at 0, verify that $\left| \int_C \left(\frac{z^2+3}{5z^3-3} \right) dz \right| \le \frac{28\pi}{37}$. Show details of your estimates.

Answer We overestimate the modulus of the integral by ML where L, the length, is (2π)radius $= (2\pi)2 = 4\pi$, and M is an overestimate of the integrand on the circle. The integrand is a quotient. We estimate it by first overestimating the top: $|z^2 + 3| \overset{\triangle}{\le} |z|^2 + 3 = 2^2 + 3 = 7$ on the circle. We next underestimate the bottom: $|5z^3 - 3| = |5z^3 + (-3)| \underset{\nabla}{\ge} |5z^3| - |-3| = 5|z|^3 - 3 = 5 \cdot 2^3 - 3 = 37$ on the circle. Therefore the modulus of the integrand is estimated by $\frac{7}{37}$ and $ML = \frac{7}{37} \cdot 4\pi = \frac{28\pi}{37}$ as desired.

(10) 9. Compute $\int_C \left(\frac{7}{z^5} + \frac{4}{z} + 5z^8 \right) dz$ where C is the boundary of the unit circle.

Answer We use linearity of the integral and evaluate the three parts separately. $\frac{7}{z^5}$ has an analytic antiderivative: use $-\frac{7}{4z^4}$ in the "punctured plane" (that is, the complex numbers except for 0). Therefore its integral around the closed curve C must be 0. The same is true for $5z^8$, where the antiderivative is $\frac{5}{9}z^9$. What about $\int_C \frac{4}{z}\, dz$? This is 4 times the value of $\int_C \frac{1}{z}\, dz = 2\pi i$. We computed this in class and it is also in the textbook. Thus the combined value of the integrals must be $4 \cdot 2\pi i = 8\pi i$.

This problem can also be done directly using a parameterization. Take $z = e^{i\theta}$ so that $dz = ie^{i\theta}\, d\theta$ with $0 \le \theta \le 2\pi$. Then $\frac{7}{z^5} + \frac{4}{z} + 5z^8$ becomes $7e^{-7i\theta} + 4e^{-i\theta} + 5e^{8i\theta}$. The whole integral is $\int_0^{2\pi} (7e^{-6i\theta} + 4e^0 + 5e^{9i\theta})\, i\, d\theta$, which can be evaluated using the Fundamental Theorem of Calculus. The $2\pi i$-periodicity of the exponential function implies that the first and third terms result in 0. The middle term is the integral of a constant on the interval from 0 to 2π. It results in $4 \cdot 2\pi \cdot i = 8\pi i$.