February 21, 2001

Review Problems for the first exam in section 5 of Math 403

Most of these are from previous Math 403 exams, and some are from the text. I have copied what's below almost exactly from a 1999 version of the course, which also has a first exam *with answers*. This can all be seen at a link from my home page (Math 403 in the spring 1999 semester), or by going directly to

`http://www.math.rutgers.edu/~greenfie/currentcourses/math403/math403_index.html`.

Realize the timing of that course and its exams was slightly different from what we are doing. Specifically, we will have covered less "integration" by the first exam (that's partly because of the snow).

I'll be in my real office (Hill 542) on Tuesday evening, February 27, from 6 PM on. (Definition of "on": if I'm alone by 7 PM and no student in the class indicates they'll be there later, I may leave. Otherwise, I will try to be available until, say, 8 PM.) If people show up and we need room to talk, we may go across the hall to Hill 525, a classroom.

1. **Elementary computations**

a) $\mathrm{Re}\left(\frac{1+i}{1-i}\right)$ b) $\mathrm{Im}\left(\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)^9\right)$ c) $\log(-1+i)$ d) $\mathrm{Log}(-e)$

2. **More computations** Solve for all possible z, expressing your answers in rectangular form:

a) $e^z=1$ b) $\cos z=0$ c) $\mathrm{Log}\ z=i$ d) $z^4=1+i$ e) $\sin z=2$

3. **Differentiability** a) Show that $g(z)=z\overline{z}$ is differentiable at 0 and is *not* differentiable at 1. In what set is g complex analytic?

b) Suppose that u and v are the real and imaginary parts of a complex analytic function, h. Suppose that the range of h always lies on the hyperbola, $uv=1$. Prove that h must be constant.

4. **Mapping** Consider the triangle T whose vertices are 0 and 1 and $1+i$. Sketch with justification the images of T under the following complex analytic functions:

a) z^2 b) $iz+2$ c) $\exp z$ d) $\mathrm{Log}\ z$* e) $\sqrt{z}$

4. **Harmonicity** a) Select constants A and B and C so that $x^3+Ax^2y+Bxy^2+Cy^3$ is harmonic, and find all harmonic conjugates of this function.

b) Verify that $\arctan\left(\dfrac{2xy}{x^2-y^2}\right)$ is harmonic**.

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* Is there a problem? Hmm ...

** Please don't do a direct computation!

5. **Elementary functions** a) Use the exponential definitions of sin and cos to verify that $\sin 2z = 2 \sin z \cos z$ for all complex z.

b) Find an example of z with $\operatorname{Arg}(z^2) \neq 2 \operatorname{Arg} z$. Give a domain of z's where the two quantities *are* equal, and explain why.

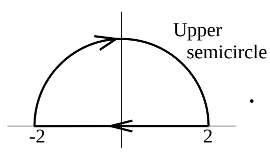
c) Suppose $H(z) = \operatorname{Log}\left(z^2 - 1\right)$. Compute $H(2)$ and $H\left(\frac{1+i}{\sqrt{2}}\right)$. Describe the largest domain containing both 2 and $\frac{1+i}{\sqrt{2}}$ in which H is complex analytic. Write explicit formulas for the real and imaginary parts of H in terms of elementary functions.

d) Describe the behavior of $\sin z$ as $z \to \infty$ along the positive imaginary axis.

e) Define the principal determination of $(z-1)^{1/2}$. Where is this function analytic?

6. **Complex contour integrals** Compute:

a) $\int_C (2x - iy)\, dz$ where C is the curve $y = x^2$ for $0 \leq x \leq 2$.

b) $\int_C |z|^2\, dz$ where C is 

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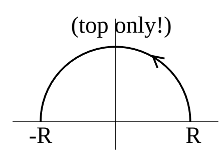
c) $\int_C z^{13} + \sin z\, dz$ where C is some simple curve from 1 to $\pi + i$.

7. **Integral estimation** a) Suppose C_R is the circle $|z| = R$ $(R > 1)$, described in the counterclockwise fashion. Show that

$$\left| \int_{C_R} \frac{\operatorname{Log} z}{z^2}\, dz \right| < 2\pi \left(\frac{\pi + \ln R}{R} \right)$$

b) Show that $\lim_{R \to \infty} \int_{C_R} \frac{z^2 - 1}{z^4 + 1}\, dz = 0$ where C_R is 

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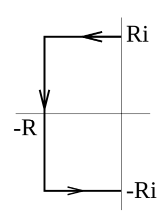
c) Show that $\lim_{R \to \infty} \int_{C_R} \frac{e^z}{z^2 - 1}\, dz = 0$ where C_R is 

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