April 8, 2001

Some answers to the review problems for the second exam in section 5 of Math 403

Please provide the instructor with corrections or comments.

1. Compute $\int_{\mathcal{C}} \frac{\sin z}{z^4}\, dz$ in several ways. For example, you could use the CIF for derivatives, or you could use the power series expansion for $\sin z$ and integrate directly.

Answer CIF for derivatives with $n = 3$ says the integral is $\frac{2\pi i}{3!}\sin^{(3)}(0) = -\frac{\pi}{3}i$. The power series for sine gives $\sin z = z - \frac{z^3}{3!} + \text{HIGHER ORDER TERMS}$, valid for all complex z, so that $\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!z} + \text{HIGHER ORDER TERMS}$. The only part giving a non-zero contribution will have a $\frac{1}{z}$ term. The integral of $\frac{1}{z}$ over $\mathcal{C}$ is $2\pi i$ so the answer this way is the same number.

2. Compute $\int_{\mathcal{C}} \frac{dz}{9+z^2}$ in several ways. For example, you could use Cauchy's Theorem, or you could find the Taylor series expansion of the integrand (via a geometric series argument) and integrate directly.

Answer The singularities of the integrand occur where $9 + z^2 = 0$: at $\pm 3i$. These are outside of $\mathcal{C}$, so that the integrand is analytic on and inside $\mathcal{C}$ and by Cauchy's Theorem the result is 0. Or we can do the following: $\frac{1}{9+z^2} = \frac{1}{9}\cdot\frac{1}{1+\frac{z^2}{9}} = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^2}{9}\right)^n$, valid where $\left|\frac{z^2}{9}\right| < 1$ (so certainly on and inside $\mathcal{C}$). But the integrals of positive integer powers of z are 0.

3. Discuss whether the following statement is always correct:

> The integral of an analytic function around a simple closed curve is always 0.

If the statement is correct, explain why it is correct. If it is not, provide some simple examples to show that it can be false. Also, give some additional hypothesis which will make the conclusion of the statement valid.

Answer The statement is false. The integral of $\frac{1}{z}$ around $\mathcal{C}$ is $2\pi i \neq 0$. The additional hypothesis "The function is analytic in a simply connected domain, and the closed curve is contained in the domain." guarantees the conclusion. A simply connected domains includes the inside of any simple closed curves it contains.

4. Evaluate the integral of $\frac{1}{(1-z)^3}$ over the following curves:

a) The circle of radius $\frac{1}{2}$ centered at 0.
b) The circle of radius $\frac{1}{2}$ centered at -1.
c) The circle of radius $\frac{1}{2}$ centered at 1.

Answer $\frac{1}{(1-z)^3}$ has antiderivative $\frac{1}{2(1-z)^2}$ in all of the complex plane except for 1. None of the closed curves given go through 1. The integrals are all 0.

5. Suppose that f is entire and you know that there is a real constant $A > 0$ so that $|f(z)| \leq A|e^z|$ for all z. Show that f must then be some constant multiple of e^z.

Hint The exponential function is never ... and things which aren't ... have multiplicative inverses – divide by them, and finish the problem with almost no effort.

Answer The function g defined by $g(z) = \frac{f(z)}{e^z}$ satisfies $|g(z)| \leq A$ for all z. Liouville's Theorem then implies that g is constant, say $g(z) = C$ for all z. Therefore $f(z) = Ce^z$ for all z.

6. Liouville's Theorem states that entire functions with "no growth" as $z \to \infty$ must be constant. In fact, functions with even "slow growth" must be constant. Please verify a version of this:

> If f is an entire function satisfying $|f(z)| \leq A\sqrt{|z|}$ for some fixed $A > 0$ and for all z with $|z|$ large, then f must be constant.

This can be done with the Cauchy inequalities, similar to the verification of Liouville's Theorem: show that $f'(z_0)$ must always be 0.

Answer Fix a complex number z_0. We estimate $|f'(z_0)$ with the first Cauchy inequality. If $R > 0$, then $|f'(z_0)| \leq \frac{2\pi}{R}\cdot(\text{SOME APPROPRIATE } M_R)$. Here the "APPROPRIATE" M_R is an overestimate of $|f(z)|$ on a circle of radius R centered at z_0. The biggest $|z|$ can be on such a circle is $|z_0| + R$ by the triangle inequality, so that $|f(z)| \leq A\sqrt{|z_0| + R}$ on the circle, and we can take M_R to be $A\sqrt{|z_0| + R}$. Combining all this we get $|f'(z_0)| \leq \frac{2\pi}{R}\cdot A\sqrt{|z_0| + R}$. But this $\to 0$ as $R \to \infty$, and the only possible value of $f'(z_0)$ is 0. Since z_0 was arbitrary, we know that f' is always 0, and therefore f must be constant.

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7. Suppose that the series $\sum_{n=0}^{\infty} a_n(3+4i)^n$ converges (here the a_n's are complex numbers). What can you then deduce about the series $\sum_{n=0}^{\infty} a_n(-1+2i)^n$? Explain your conclusion.

Answer $|3+4i| = 5$ and $|-1+2i| = \sqrt{5}$. The series $\sum_{n=0}^{\infty} a_n z^n$, a power series "centered" at 0, converges at $3+4i$. At points closer to 0 than $3+4i$ the series must converge, and actually converge absolutely. Since $\sqrt{5} < 5$, we have two conclusions: $\sum_{n=0}^{\infty} a_n(-1+2i)^n$ converges absolutely, and it therefore also converges.

8. What is the radius of convergence of the Taylor series expansion of $f(z) = \frac{e^z}{(z-1)(z+1)(z-2)(z-3)}$ when expanded around $z = i$? Give a numerical answer. Justify why the series must converge with at least that radius and why it can't have a larger radius. I don't think that actual computation of the series is practical!

Answer f is analytic everywhere except 1, -1, 2, and 3. We know that the Taylor series centered at z_0 of an analytic function must converge at least in the largest disc centered at z_0 in which the function is analytic. In this case the center z_0 is i. The closest singularity is 1 or -1. The radius of convergence must be at least $\sqrt{2}$, the distance from i to one of ± 1. Why can't the series have a larger radius of convergence? If it did, then the resulting sum would represent an analytic function. Let's call that function, g. Actually g and f must coincide for $|z-i| < \sqrt{2}$ because the series is the Taylor series for f centered at i and we know that such series must converge to the original function. But what can $g(1)$ be? Certainly g should be continuous, so that $\lim_{r\to 1^-} g(i + r(1-i)) = g(1)$. This is a precise way of writing the values of g on the line segment joining i and 1. But $g(i+r(1-i)) = f(i+r(1-i))$, and the modulus of $f(i+r(1-i))$ approaches ∞ as $r \to 1^-$ (examine the formula for f). So the radius of convergence can't be larger than $\sqrt{2}$.

9. Find two distinct Laurent series for the function $g(z) = \frac{1+3z}{4z^2+z^4}$ and in each case specify the annulus in which the series is valid. Find explicit values of the coefficients of z^{-36} and z^{35} in each case.

Answer We'll look for Laurent series centered at 0. Prepare the formula for g a little bit: $\frac{1+3z}{4z^2+z^4} = \frac{1+3z}{z^2} \cdot \frac{1}{4+z^2}$. There are two cases: $|z| < 2$ and $|z| > 2$.

For the first, $\frac{1}{4+z^2} = \frac{1}{4} \cdot \frac{1}{1+\frac{z^2}{4}} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^2}{4}\right)^n$. So: in the annulus $0 < |z| < 2$, $g(z) = \frac{1+3z}{z^2} \cdot \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z^2}{4}\right)^n$. There is "no" z^{-36}, so the coefficient of z^{-36} is 0. There are two possible contributions to the z^{35} term: $\frac{1}{z^2} \cdot \frac{1}{4} \cdot (-1)^n\left(\frac{z^2}{4}\right)^n$ – but this only has even powers of z, and $\frac{3z}{z^2} \cdot \frac{1}{4} \cdot (-1)^n\left(\frac{z^2}{4}\right)^n$, which gives z^{35} when $n = 18$. The coefficient must be $\frac{3}{4} \cdot (-1)^{18} \cdot 4^{-18}$.

If $|z| > 2$, $\frac{1}{4+z^2} = \frac{1}{z^2} \cdot \frac{1}{1+\frac{4}{z^2}} = \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{z^2}\right)^n$. If $2 < |z| < \infty$, $g(z) = \frac{1+3z}{z^2} \cdot \frac{1}{z^2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{4}{z^2}\right)^n$. Here there is no z^{35} term. There are two possible contributions to the z^{-36} term: $\frac{1}{z^2} \cdot (-1)^n\left(\frac{4}{z^2}\right)^n$ which when $n = 17$ is $(-1)^{17}4^{17}z^{-36}$ and the other, gotten by multiplying the sum by $\frac{3z}{z^2}$, has only odd degree terms. The coefficient must be $(-1)^{17}4^{17}$.

10. a) Find a Laurent series for $e^{-\frac{1}{z}}$ and describe those z's for which it is valid.

Answer Since $e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$ is true for all complex z, we know that when $z \neq 0$, $e^{-\frac{1}{z}} = \sum_{n=0}^{\infty}\frac{(-1)^n z^{-n}}{n!}$ is true. This Laurent series is correct in the annulus $0 < |z| < \infty$.

b) Use your answer to a) to compute $\int_{\mathcal{C}} z^4 e^{-\frac{1}{z}}\,dz$.

Answer Since $\mathcal{C}$ is inside the annulus described in a) we can use the expansion given in a) to compute the integral. $z^4 e^{-\frac{1}{z}} = z^4 \cdot \sum_{n=0}^{\infty}\frac{(-1)^n z^{-n}}{n!}$. The only term giving a non-zero integral is a $\frac{1}{z}$ term, and this occurs when $n = 5$: $z^4 \cdot \frac{(-1)^n z^{-5}}{5!} = -\frac{1}{5!z}$. The integral is $2\pi i \cdot -\frac{1}{5!}$.

11. a) If h is defined by $h(z) = \frac{\sin(z^2)}{2+z^3}$ find the terms up to and including degree 5 in the Taylor series centered at 0 of h.

Answer Since $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!}$+HIGHER ORDER TERMS, we know $\sin(z^2) = z^2 - \frac{z^6}{3!}$+HIGHER ORDER TERMS. Also $\frac{1}{2+z^3} = \frac{1}{2} \cdot \frac{1}{1+\frac{z^3}{2}} = \frac{1}{2}\left(1 - \frac{z^3}{2} + \frac{z^6}{4}\right.$+HIGHER ORDER TERMS$\left.\right)$. Now multiply, but keep only terms with degree ≤ 5. The result is $\frac{1}{2}z^2 - \frac{1}{4}z^5$.

b) Use your answer to a) to compute $h^{(5)}(0)$.

Answer The coefficient of z^5 must be $\frac{h^{(5)}(0)}{5!}$. So $h^{(5)}(0) = -\frac{1}{4} \cdot 5! = -30$.